

Absolute magnetic field sensing with a resonant double SQUID circuit

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Abstract

Superconducting quantum interference devices (SQUIDs) are among the most precise magnetic-field sensors available and are widely used for this purpose. However, single SQUIDs face one major challenge: while they are very precise when measuring small changes in magnetic field, their periodic response to magnetic flux prevents the direct measurement of the absolute value of the magnetic field.

The aim of this work is to overcome this limitation. The idea is to combine two SQUID loops with different areas in a single resonant circuit, thereby producing a large synthetic period. A change in magnetic field is reflected as a change in the resonance frequency of the circuit. For suitably chosen loop sizes, this leads to an effective magnetic field period which is orders of magnitude larger than that of each individual loop. This allows absolute field measurements when sufficient prior knowledge of the field is available.

We investigated the framework of such a system, analysed design concepts, and fabricated two prototypes. Experimental characterisation of the first prototype revealed that sensitivity to magnetic field gradients becomes a major issue for determining the offset field. The sensitivity to magnetic field gradients becomes larger when extending the magnetic-field period by larger factors. For the second prototype, we achieved an extended period of greater than $2.8 \mu\text{T}$, which is about 45 times the period of the larger SQUID loop, while still keeping the system very resistant to gradient-induced errors.

We further analysed the shielding performance of a double-layer Mu-metal shield and measured a shielding attenuation of $48.0(4) \text{ dB}$. The magnetic-field noise behind the same shielding was characterised by fitting $1/f$ noise plus a white noise amplitude S_0 . We could determine the white noise amplitude to be $S_{0,\text{day}}^{1/2} = 13.8 \mu\Phi_0/\sqrt{\text{Hz}}$ during the day and $S_{0,\text{night}}^{1/2} = 10.0 \mu\Phi_0/\sqrt{\text{Hz}}$ during the night.

We could also determine the offset field behind a single layer Mu-metal shield to be $2.40(3) \mu\text{T}$. This value is approximately a factor of five higher than the residual field measured at room temperature. Therefore, the measurement is not conclusive enough to confirm the working principle of the device.

Kurzfassung

SQUIDs zählen zu den genauesten und präzisesten Magnetfeldsensoren und sind bereits weit verbreitet. Die Abkürzung stammt aus dem Englischen und steht für *superconducting quantum interference device*. Diese Sensoren ermöglichen es, Änderungen im Magnetfeld mit sehr hoher Genauigkeit zu bestimmen. Dabei besteht jedoch eine grundlegende Einschränkung: Das messbare Signal eines SQUIDs ist inhärent periodisch. Diese Periodizität verhindert, dass der Absolutwert des Magnetfeldes direkt bestimmt werden kann.

Ziel dieser Arbeit ist es, diese Limitation zu überwinden. Die zugrunde liegende Idee besteht darin, zwei SQUID-Schleifen in einem elektrischen Schwingkreis zu kombinieren. Eine Änderung des Magnetfeldes führt dadurch zu einer Änderung der Resonanzfrequenz des Schaltkreises. Werden die Größen der SQUID-Schleifen – und damit die jeweilige Magnetfeldperiode – geeignet gewählt, kann die Gesamtperiode des Systems deutlich erhöht werden. Diese kann mehrere Größenordnungen über der Periode eines einzelnen SQUIDs liegen und ermöglicht somit die Bestimmung des Absolutwertes des Magnetfeldes, sofern dieser im Vorfeld ausreichend genau bekannt ist.

Wir haben das Verhalten solcher Doppel-SQUID-Systeme untersucht und zwei Prototypen entworfen und gefertigt. Experimentelle Messungen am ersten Prototypen zeigten, dass die Bestimmung des Absolutfeldes sehr sensitiv gegenüber Gradienten im Magnetfeld ist. Die Unsicherheit durch Gradienten nimmt zu, je größer die Gesamtperiode des Systems im Vergleich zur Periode eines einzelnen SQUIDs gewählt wird. Mit dem zweiten Prototypen konnte eine Periodizität von $2.8\ \mu\text{T}$ erreicht werden, was etwa dem 45-fachen der Einzelperiode der größeren SQUID-Schleife entspricht. Gleichzeitig wurde die Sensitivität gegenüber Gradienten stark eingeschränkt.

Darüber hinaus wurde das Verhalten einer doppelagigen Mu-Metall-Abschirmung untersucht. Die Abschirmung gegenüber äußeren Magnetfeldern konnte dabei zu $48.0(4)\ \text{dB}$ bestimmt werden. Zusätzlich wurde das Magnetfeldrauschen hinter der Abschirmung gemessen und mit einem Rauschmodell bestehend aus $1/f$ -Rauschen und einer Weißrausch-Amplitude S_0 charakterisiert. Daraus ergaben sich Amplituden für das Weißrauschen von $S_{0,\text{day}}^{1/2} = 13.8\ \mu\Phi_0/\sqrt{\text{Hz}}$ am Tag sowie $S_{0,\text{night}}^{1/2} = 10.0\ \mu\Phi_0/\sqrt{\text{Hz}}$ in der Nacht.

Zusätzlich konnte das Absolutfeld hinter einer einlagigen Mu-Metall-Abschirmung zu $2.40(3)\ \mu\text{T}$ bestimmt werden. Dieser Wert liegt etwa um einen Faktor fünf über dem bei Raumtemperatur bestimmten Restfeld. Die Messung ist daher nicht ausreichend konsistent, um das Funktionsprinzip des Sensors eindeutig zu bestätigen.

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Contents

Contents	viii
1 Introduction	1
2 Theoretical background	5
2.1 Classical Circuits	6
2.1.1 Circuit Basics	6
2.1.2 Circuit Elements	7
2.1.3 LC-Oscillator	8
2.1.4 Circuit Impedance	8
2.2 Superconductivity and Josephson Junctions	9
2.2.1 Superconductivity	9
2.2.2 Josephson Junction	12
2.2.3 Quantum Mechanical Oscillator	14
2.3 SQUIDS	16
2.3.1 SQUIDS without geometrical inductance	16
2.3.2 SQUIDS with geometrical inductance	18
3 Absolute field sensing: Double SQUID circuit	23
3.1 Loop area ratio and absolute field measurement	24
3.1.1 Limitations	26
3.1.2 Effects of field inhomogeneity	27
3.2 Two SQUIDS in series	31
3.2.1 Simplification 1: No geometric inductances	32
3.2.2 Simplification 2: With geometric inductances	33
3.2.3 Fitting function and parameters	36
3.3 Prototype design goals	38
4 Fabrication	43
4.1 Fabrication techniques	43
4.2 Machines	45
4.3 Junction fabrication	48
4.3.1 Junction Parameters	48
4.4 Settings and Recipes	52
5 Measurements and Characterisation	55
5.1 Experimental setup	55
5.1.1 Cryostat and wiring	56

5.1.2	Sample mounting and DC coils	57
5.2	Characterisation	58
5.2.1	Flux maps	59
5.2.2	Quality Factors	63
5.2.3	Power behaviour	66
5.2.4	Hysteresis	70
5.3	Magnetic field measurements	74
5.3.1	Offset field for different sample shielding configurations	74
5.3.2	Attenuation of the shield	79
5.3.3	Magnetic field change over time	81
6	Conclusion and Outlook	87
7	Appendix	91
7.1	Solution of the circuit model with linear inductance	91
	Bibliography	95

CHAPTER 1

Introduction

Long before people could explain magnetic fields mathematically, the existence of magnetic fields was known and utilized. Measuring the direction of the Earth's magnetic field with a compass improved navigation and allowed journeys to far distant places. Since then, significant technological progress has been made, and today the applications for precise magnetic-field sensors, called magnetometers, are enormous. They are used, for example, in geological surveys for ore detection, nondestructive material testing, and magnetoencephalography for mapping brain activity, to name just three of numerous technical applications [1].

Besides these, there is also a huge need for magnetic-field sensing in fundamental physics applications. Especially quantum physics experiments can be sensitive to external magnetic fields. Atomic transition frequencies, for example, depend on the Zeeman effect and thus on the amplitude of magnetic fields along the quantization axis [2]. Experiments working with qubits in the form of spin eigenstates in nitrogen-vacancy defects in diamonds are also sensitive to magnetic-field noise [3]. Another platform where magnetic-field noise plays an essential role is superconducting quantum computing. Here, it is one of the major sources of decoherence for various qubit architectures [4, 5].

The huge demand for magnetometers has led to more and more precise measurement methods, with superconducting interference devices (SQUIDs) being among the most precise magnetic-field sensors available. The advantage of a SQUID is that the sensitivity to magnetic fields is proportional to the SQUID area, allowing nearly arbitrarily high sensitivity only limited by size limitations due to fabrication. The greatest strength of a SQUID is at the same time also a downside: the periodic response of the system does not allow the measurement of absolute magnetic fields, while changes in magnetic field can be measured very accurately.

The principal idea to overcome this limitation is to combine multiple SQUID loops (with different

areas) in one system in order to obtain an extended period. There are multiple concepts for how this can be done.

One idea, for example, is the use of multiple DC-SQUIDs in series as in Ref. [6]. In this work, seven SQUID loops are used. When biased with a constant current, the voltage across a SQUID is a function of the magnetic field threading it. In the case of zero magnetic field, every loop will be in the superconducting phase, which leads to a minimum voltage drop. One shortcoming of this idea is that the SQUID loops will have different properties and thus different optimal operation points. Another problem could arise when the magnetic fields through the loops differ too much; in that case the global voltage minimum would disappear. Such a system can be extended to longer arrays of SQUID loops, as demonstrated in Ref. [7] with 211 SQUID loops. Here, a pickup loop was used to focus the external field and achieve higher sensitivity. At the position of zero field, a distinct drop in voltage was reported.

Another concept is based on combining three SQUIDs with strongly different effective areas, as demonstrated in Ref. [8]. In this approach, the periodic ambiguity of a SQUID is resolved by using additional reference SQUIDs with smaller effective areas, which uniquely determine the operating branch of a more sensitive SQUID.

A similar approach is to measure the voltage drop across a 2D array of SQUID loops with identical size as done in this relatively new work [9]. By including bare superconducting loops without Josephson junctions, the response of the system is no longer periodic and thus allows absolute field measurement.

These were three examples where the SQUID is operated in the dissipative regime, where the voltage drop across a DC-biased device is measured. Alternatively, a SQUID can be operated in the dispersive regime. In this case, changes in the SQUID inductance induced by an external magnetic field are detected. This approach enables magnetic-field measurements with high bandwidth while introducing only minimal backaction on the environment.

Absolute field measurement in the dispersive regime is demonstrated in Ref. [10]. In this work the inductance of two SQUID loops is mapped onto the resonance frequency of a coupled circuit. Our work closely follows the principle proposed in [10].

The advantage of this approach is that a system consisting of only two SQUID loops can be understood more intuitively than large arrays of SQUIDs. When choosing the area ratio of the loops wisely, the magnetic period of the system can be sufficiently enhanced. The signal will still be periodic, but the period can be extended to the point where it effectively allows absolute measurements when sufficient prior knowledge of the ambient magnetic field is available.

The main contributions of this work are:

- Developing design concepts for such a double-SQUID system and analysing the underlying limitations for the design parameters
- Fabrication of two prototypes with different designs
- Characterisation of these prototypes and investigation of their behaviour

- Using the prototypes to perform measurements that characterise different magnetic shields in the laboratory

This thesis is organised as follows. In chapter 2 we present background knowledge relevant to microwave superconducting circuits. It is intended to help understand the design considerations and measurements presented in the later chapters.

In chapter 3 we begin by discussing the general operating principle of a system consisting of two SQUID loops and analyse the fundamental limitations of this approach. Subsequently, we introduce a specific implementation of a double-SQUID system based on transmon-like antenna pads. Finally, we examine the two fabricated prototypes and motivate the corresponding design choices based on the behaviour of the general system.

In chapter 4 we explain the fabrication process. This chapter serves both as a short introduction to the fabrication techniques used and as a guideline for anyone who intends to fabricate a similar device.

In chapter 5 we present all measured aspects of both prototypes. The chapter begins with a description of the experimental setup. We then show the device responses and extract their characteristic parameters. Afterwards, we address the primary purpose of the device, namely magnetic-field sensing. This includes offset-field measurements, magnetic-shield attenuation, and noise characterisation.

In the final chapter 6 we draw conclusions from the measurements and discuss possible next steps for the project.

CHAPTER 2

Theoretical background

In this chapter, we work through the theoretical background necessary for understanding the design considerations and measurements presented in this thesis. The theory is intentionally kept concise, with references to relevant literature for readers seeking more detailed information.

The first part of the chapter focuses on classical circuits. We introduce capacitors and inductors as basic circuit elements which, when combined, form the simplest version of an electrical oscillator. This section concludes with the concept of circuit impedance, providing a framework for analysing more complex electrical circuits.

We then move beyond classical physics and examine the Josephson junction. Starting from the fundamentals of superconductivity, we discuss the behaviour of the junction and how it operates within a circuit.

In the final part, we explore how two Josephson junctions can be combined to form a SQUID loop. We discuss its various properties and analyse its response to applied magnetic fields for different design parameters.

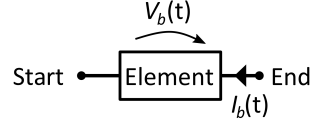


Fig. 2.1 Example branch b with voltage $V_b(t)$ across it and current $I_b(t)$ flowing through it, connecting the start and end node. Current and voltage are defined in opposite directions. Figure taken from [11].

2.1 Classical Circuits

In the following, we consider electrical circuits with lumped inductances and capacitances. We will later build upon this in the non-linear regime. Although the circuits discussed in this thesis are in general non-linear, they can often be simplified to the linear case, which significantly eases their description.

2.1.1 Circuit Basics

In this section, we briefly review the basic elements of superconducting circuits. The presentation follows the work of [11], where interested readers can find more details.

To describe and characterise a superconducting circuit, the system can be represented with a lumped-element circuit diagram. In this model, a spatially distributed element (e.g. a long, thin superconducting wire) is represented as a discrete element with certain characteristics (e.g. an inductive element). These discrete elements are connected in a network by perfectly conducting wires, as illustrated in Fig. 3.3. To characterise such a system, it is necessary to determine the current and voltage across each element. All elements considered here have only two terminals; we call such a connection a *branch*, with a voltage drop $V_b(t)$ across it and a current $I_b(t)$ flowing through it. The endpoints of a branch are called *nodes*. A single element is shown in Fig. 2.1.

For the description of the system, it is useful to introduce the integrated variables: the branch flux $\Phi_b(t)$ and the charge $Q_b(t)$, defined as

$$\Phi_b(t) = \int_{-\infty}^t V_b(t') dt', \quad (2.1a)$$

$$Q_b(t) = \int_{-\infty}^t I_b(t') dt'. \quad (2.1b)$$

The lower integration limit is set to $t_0 = -\infty$, where the system is assumed to be at rest, i.e. no voltages or currents are present. These integrated variables are constrained by Kirchhoff's laws. The first law,

$$\sum_{\substack{\text{all } b \\ \text{arriving at } n}} Q_b = Q_n, \quad (2.2a)$$

states that no net charge can accumulate at a node and is equivalent to the continuity condition for the current. The second law,

$$\sum_{\substack{\text{all } b \\ \text{around } l}} \Phi_b = \tilde{\Phi}_l, \quad (2.2b)$$

requires that the sum of all branch fluxes around a closed loop equals the applied external magnetic flux $\tilde{\Phi}_l$ threading the loop. These constraints reduce the number of free degrees of freedom in the circuit.

2.1.2 Circuit Elements

Relevant for the circuits considered in this work are three components: inductances L , capacitors C , and Josephson junctions (JJ). Since we work with superconducting circuits, we can neglect resistive losses when analysing resonance frequencies and mode structures, and therefore we omit resistive elements. Josephson junctions can be viewed as a special type of inductor with non-linear behaviour and are thus beyond the scope of the classical description. Their properties will be discussed in detail in section 2.2.2. Here, we briefly describe the linear components.

Capacitor

A capacitor is characterised by the relation between stored charge and applied voltage. In general, the charge may be an arbitrary function of the voltage, $q(t) = f[V(t)]$. Fortunately, capacitances relevant in our context are linear and are therefore well described by

$$V(t) = \dot{\Phi}(t) = \frac{1}{C}q(t). \quad (2.3)$$

The charge is proportional to the voltage, with the ratio given by the capacitance C . Equation (2.3) also directly expresses the flux–charge relation of a capacitor $I(t) = C\ddot{\Phi}(t)$. The energy stored in a capacitor can be obtained by summing the work needed to accumulate charge on its plates:

$$E = \int_0^t P(t) dt = \int_0^t v(t) i(t) dt = \int_0^t v(t) \frac{dq}{dt} dt = \int_0^Q \frac{q}{C} dq = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C \dot{\Phi}^2. \quad (2.4)$$

The energy of a capacitor is stored in the electromagnetic field between two conductors of arbitrary shape. This holds in vacuum or air, but the effect can be enhanced if the electric field (partially) passes through an insulating dielectric. Choosing a suitable dielectric can increase the capacitance by an order of magnitude compared to air.

Inductor

An inductor is characterised by the relation between applied flux and current, generally written as $I(t) = f[\Phi(t)]$. As before, we restrict ourselves to linear inductors, described by

$$I(t) = \dot{q}(t) = \frac{1}{L}\Phi(t). \quad (2.5)$$

The current is proportional to the applied flux, with the proportionality set by the inductance L . The stored energy is given by

$$E = \int_0^t P(t) dt = \int_0^t i(t) v(t) dt = \int_0^t i(t) L \frac{di}{dt} dt = \int_0^I i L di = \frac{1}{2} LI^2, \quad (2.6)$$

where I is the maximum current through the inductor. The energy is stored in the magnetic field generated by the current. In general, every current through any wire generates a magnetic field described by the Biot–Savart law. The effect is enhanced when these wires are wound into coils enabling long wire lengths in small space, as commonly done in classical engineering devices. Similarly, a straight (superconducting) wire has some inductance, the smaller the conductor diameter and the longer the wire, the larger its inductance.

2.1.3 LC-Oscillator

Capacitors and inductors can be used to build electrical oscillators. To illustrate this, let us consider the simplest model: one inductor with inductance L and one capacitor with capacitance C connected in a closed circuit, as shown in Fig. 2.5. The total energy of the system is

$$H = \frac{LI^2}{2} + \frac{CU^2}{2} = \frac{\Phi^2}{2L} + \frac{Q^2}{2C}, \quad (2.7)$$

where U is the voltage across the capacitor and I the current through the inductor. In the second equality we express the Hamiltonian in terms of the charge Q and flux Φ variables.

Making an analogy to a mechanical mass–spring oscillator, we see that Φ plays the role of a canonical position variable, with L determining the strength of the restoring force. Conversely, the charge Q corresponds to the canonical momentum variable, with C acting as an effective mass. The dynamics of the system follows from solving the equation of motion, leading to oscillatory behaviour in which energy is exchanged between the inductor and capacitor. The oscillation frequency is

$$\omega = \frac{1}{\sqrt{LC}}.$$

In the following section, we discuss how resonance frequencies of more complex circuits can be obtained using a simple impedance-based framework.

2.1.4 Circuit Impedance

Impedance is a fundamental concept in electrical engineering that generalises resistance to alternating-current (AC) circuits. The framework arises naturally from applying a Fourier transform to the equation of motion, converting differential equations into algebraic ones. While resistance characterises opposition to direct current (DC), impedance accounts for both resistive and reactive (energy-storing) properties of inductors and capacitors.

The impedance $\bar{Z}$ of a single element is expressed as the complex ratio

$$\bar{Z} = \frac{\bar{u}}{\bar{i}}, \quad (2.8)$$

where $\bar{u}$ is the complex representation of the sinusoidal voltage and $\bar{i}$ the complex current at a given time when driving the circuit with a sinusoidal signal of frequency ω . Such a description is valid for a driven network in steady state, which is sufficient for our use case. The impedances of a resistor, capacitor, and inductor are

$$Z_R = R, \quad (2.9a)$$

$$Z_L = i\omega L, \quad (2.9b)$$

$$Z_C = \frac{1}{i\omega C} = -\frac{i}{\omega C}, \quad (2.9c)$$

where i is the imaginary unit. These expressions reflect the fact that the current through a capacitor or inductor is phase-shifted by 90° relative to the voltage. For a capacitor, the current leads the voltage; for an inductor, the voltage leads the current. A resistor exhibits no phase shift and therefore has a purely real impedance.

These relations also show that a capacitor behaves like an infinite impedance at DC ($\omega \rightarrow 0$), whereas an inductor behaves like an infinite impedance at very high frequencies ($\omega \rightarrow \infty$).

So far we have discussed single lumped elements, but the impedance formalism allows the analysis of more complex networks. Impedances combine in direct analogy to resistances

$$Z_{\text{parallel}}^{-1} = Z_1^{-1} + Z_2^{-1} + Z_3^{-1} + \dots, \quad (2.10a)$$

$$Z_{\text{series}} = Z_1 + Z_2 + Z_3 + \dots. \quad (2.10b)$$

The resonance frequencies of such circuits can be found by identifying the frequencies at which the imaginary part of the total impedance vanishes. For circuits without resistive elements—as is the case in our later models—the resonance frequencies are obtained by solving for the zeros of the impedance. This yields analytical expressions for the eigenfrequencies of classical circuits but does not capture effects arising from external magnetic flux or non-linear elements.

2.2 Superconductivity and Josephson Junctions

Superconductivity is the essential phenomenon behind the magnetic-field sensor used in this work. It explains the behaviour of Josephson junctions and the quantisation of magnetic flux in closed loops. In this section, we briefly review the fundamentals of superconductivity; the interested reader is referred to [12] for a more complete description.

2.2.1 Superconductivity

Superconductivity was discovered by H. Kamerlingh Onnes in 1911 during resistance measurements on mercury cooled with a then-new helium refrigeration system. He observed that the resistance dropped to zero when mercury was cooled below a critical temperature T_c [13]. A table of critical temperatures for selected materials is given later in this chapter in Tab. 2.1.

The origin of superconductivity lies in an effective attractive interaction between electrons, mediated by phonons of the atomic lattice. Electrons pair up to form Cooper pairs with opposite momentum and opposite spin. This pairing overcomes the fermionic exclusion principle and allows the Cooper pairs to condense into a collective bosonic ground state. The energy gap between this superconducting ground state and the Fermi energy is

$$\Delta = 1.76 k_B T_c,$$

with the Boltzmann constant $k_B \approx 1.38 \times 10^{-23}$ J/K. The critical temperature T_c is material dependent. Many pure elements become superconducting when cooled sufficiently, as do numerous alloys and ceramics [12]. The superconducting state is well described by BCS theory (Bardeen, Cooper, Schrieffer). In this work, however, we focus on simpler considerations based on London theory and Ginzburg–Landau theory. Both of these theories were derived phenomenologically: they do not explain *why* a material becomes superconducting, but they provide an accurate description of *how* superconductors behave. The macroscopic state of a superconductor is characterised by the Cooper-pair density $n_s(\vec{r}, t)$ and a superconducting phase $\Theta(\vec{r}, t)$, forming the macroscopic wavefunction [12]

$$\Psi(\vec{r}, t) = \sqrt{n_s(\vec{r}, t)} e^{i\Theta(\vec{r}, t)}. \quad (2.11)$$

Inserting this wavefunction into the Schrödinger equation and performing several transformations yields the London relation

$$\Lambda \vec{J}_s(\vec{r}, t) = -\vec{A}(\vec{r}, t) - \frac{\hbar}{2e} \nabla \Theta(\vec{r}, t), \quad (2.12)$$

which describes the supercurrent $\vec{J}_s$ in terms of the vector potential $\vec{A}$ and the phase gradient. The parameter

$$\Lambda = \frac{m_e}{2\bar{n}_s e^2}$$

is the London parameter and depends on the mean Cooper-pair density $\bar{n}_s$. Equation (2.12) shows that supercurrents can be generated either by a vector potential $\vec{A}$ and thus magnetic fields through $\nabla \times \vec{A} = \vec{B}$, or by gradients of the superconducting phase Θ , related to the electric field via $\vec{E} = \partial \vec{A} / \partial t - \nabla \Theta$.

A striking feature of superconductors is their ability to expel magnetic fields. Taking the curl of Eq. (2.12) yields

$$\nabla \times (\Lambda \vec{J}_s) = -\vec{B}. \quad (2.13)$$

This is the second London equation and shows that a magnetic field inside a superconductor induces screening currents orthogonal to the magnetic field. Taking a second curl leads—after neglecting small terms—to the differential equation

$$\nabla^2 \vec{B} = \frac{1}{\Lambda_L^2} \vec{B}, \quad (2.14)$$

where ∇^2 is the Laplacian. Solving this equation reveals that magnetic fields decay exponentially inside a superconductor. The characteristic decay length is the London penetration depth,

$$\Lambda_L = \sqrt{\frac{\Lambda}{\mu_0}} = \sqrt{\frac{m_e}{2\mu_0 \bar{n}_s e^2}}. \quad (2.15)$$

The penetration depth depends on the material and also on the thickness of the superconducting film. The expulsion of magnetic fields occurs both when a magnetic field is applied after cooling below T_c , and when cooling in an already present magnetic field. Thus, a superconductor is not only a perfect conductor but also a perfect diamagnet. This phenomenon is known as the Meißner–Ochsenfeld effect. The resulting shielding currents and magnetic-field decay are illustrated in Fig. 2.2. The Meißner effect holds only up to a certain field strength. The energy required to generate screening currents is limited by the condensation energy gained in the superconducting phase transition. Exceeding this limit destroys superconductivity. Two types of superconductors exist, distinguished by their behaviour at high magnetic fields. The classification arises from Ginzburg–Landau (GL) theory [14], which treats superconductivity thermodynamically.

In GL theory, the coherence length ζ_{GL} characterises the length scale over which the superconducting wavefunction recovers from perturbations. Combining the coherence length with the London penetration depth yields the Ginzburg–Landau parameter

$$\kappa_{GL} = \frac{\Lambda_L}{\zeta_{GL}}. \quad (2.16)$$

This parameter distinguishes Type I and Type II superconductors:

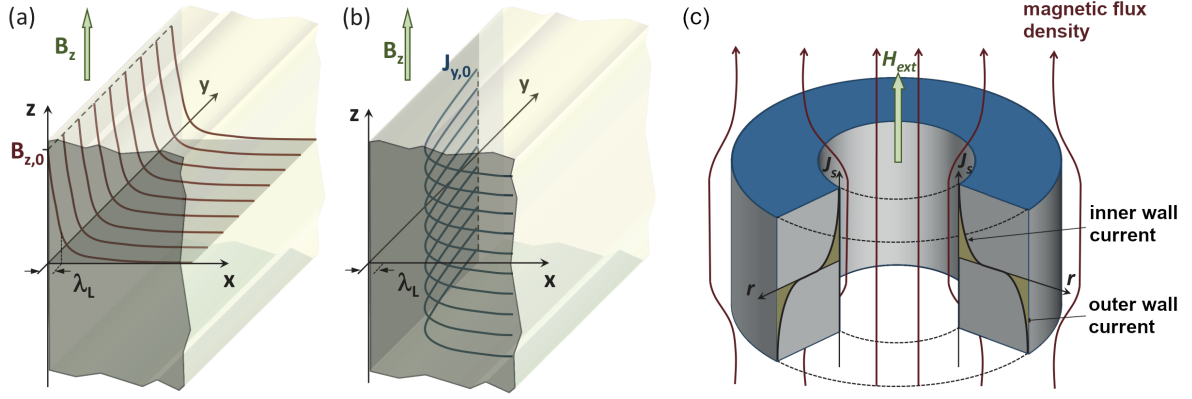


Fig. 2.2 Magnetic field shielding of a superconductor. (a) Exponential decay of the magnetic flux density B . (b) Superconducting screening current J_s as a function of the distance x from the surface of a bulk superconductor. The external field is applied in the z -direction, and the superconductor occupies the half-space $x \geq 0$. (c) Schematic illustration of a hollow superconducting cylinder that has been cooled below its critical temperature while placed in a constant external magnetic field H_{ext} . Shown are the magnetic field lines as well as the expected distribution of the superconducting screening current, which flows as a ring current on the surface of the cylinder wall. The opposite signs of J_s on the inner and outer surfaces indicate the opposite rotation of these ring currents. Figure and caption adapted from [12].

- **Type I superconductors**

Expel magnetic fields completely (Meißner state) up to a critical field B_c . This occurs for $\kappa_{GL} < 1/\sqrt{2}$. For stronger magnetic fields, superconductivity breaks down and the material becomes normal conducting.

- **Type II superconductors**

Occur for $\kappa_{GL} > 1/\sqrt{2}$. They exhibit two critical fields. For fields below B_{c1} , the behaviour is identical to Type I superconductors. For $B_{c1} < B < B_{c2}$, magnetic flux penetrates the superconductor in the form of small normal-conducting tubes, called vortices, as shown in Fig. 2.3. This mixed state is energetically favourable when the penetration depth λ_L is large compared to the coherence length ξ_{GL} . The number of vortices increases with field strength until superconductivity collapses at B_{c2} .

Material	Type	T_c (K)	$\Delta(0)$ (meV)	B_c (mT)	λ_L (nm)	ξ_{GL} (nm)
Al	I	1.19	0.180	10.5	50	1550
Ta	I	4.47	0.650	82.9	35	~ 100
Nb	II	9.25	1.45	206	32–45	39

Tab. 2.1 Superconducting parameters for selected materials. Values taken from [12, 15].

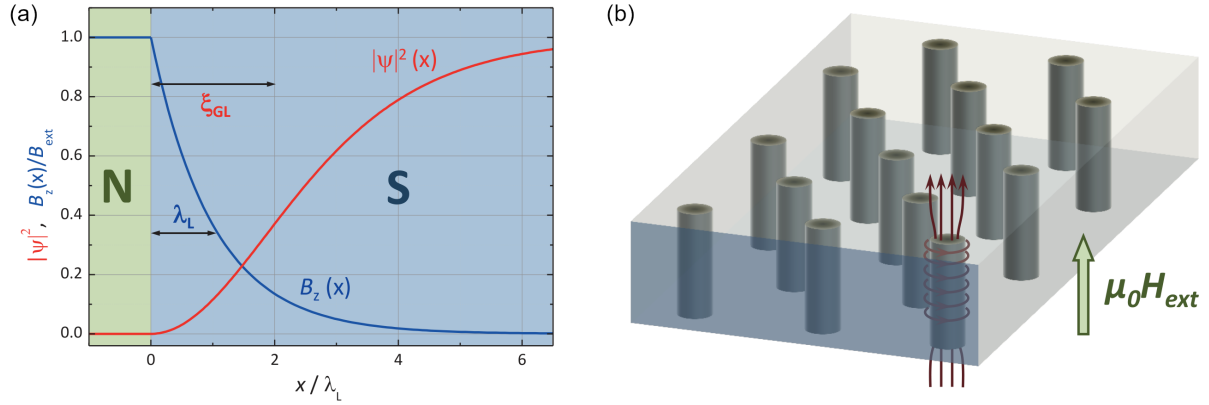


Fig. 2.3 (a) Profile of the absolute square of the normalised order parameter $\psi(x) = \psi(x)/\psi_0$ at a normal-metal–superconductor interface. The superconductor occupies the half-space $x \geq 0$. A value of $\zeta_{GL} = 2\lambda_L$ is assumed. For comparison, the decay of an external magnetic field B_{ext} applied in the z -direction is shown. (b) Schematic illustration of the flux-line lattice in a type-II superconductor. For a single flux line, the magnetic-field profile and the circulating screening currents are sketched. Figure and caption adapted from [12].

Flux Quantisation

One fundamental property of superconductors is the quantisation of magnetic flux. If an area is completely surrounded by a superconducting material, the magnetic flux through this area can only take discrete values. This can be illustrated by integrating Eq. 2.12 along a closed contour inside the superconducting region. Since the phase of the wavefunction must be single valued, we have to return to its original value after completing a loop, its change can only be an integer multiple of 2π . This leads to the relation

$$\oint_{\partial S} \Lambda \vec{J}_s d\vec{l} + \int_S \vec{B} d\vec{s} = \oint_{\partial S} \Lambda \vec{J}_s d\vec{l} + \Phi_{loop} = n\Phi_0, \quad (2.17)$$

where Φ_0 is the magnetic flux quantum and n is an integer. The left side consists of the line integral over the supercurrent density along the contour ∂S and the magnetic flux Φ_{loop} threading the enclosed area S .

Let us now consider a superconducting ring, as indicated in Fig. 2.2, with a wall thickness much larger than the London penetration depth ($d \gg \lambda_L$). If we choose a contour line deep inside the superconductor, the supercurrents vanish, simplifying Eq. 2.17 and showing that the total flux through the loop is quantised. This total flux includes both the externally applied magnetic field and the field generated by induced shielding currents on the inner surface of the ring. As long as superconductivity persists, flux quantisation remains valid, independent of variations in the external field. A similar argument holds for vortices in type-II superconductors, where each vortex carries an integer number of flux quanta.

2.2.2 Josephson Junction

A Josephson junction is a thin non-superconducting barrier between two superconductors. The barrier must be sufficiently thin so that the superconducting wavefunctions on both sides overlap

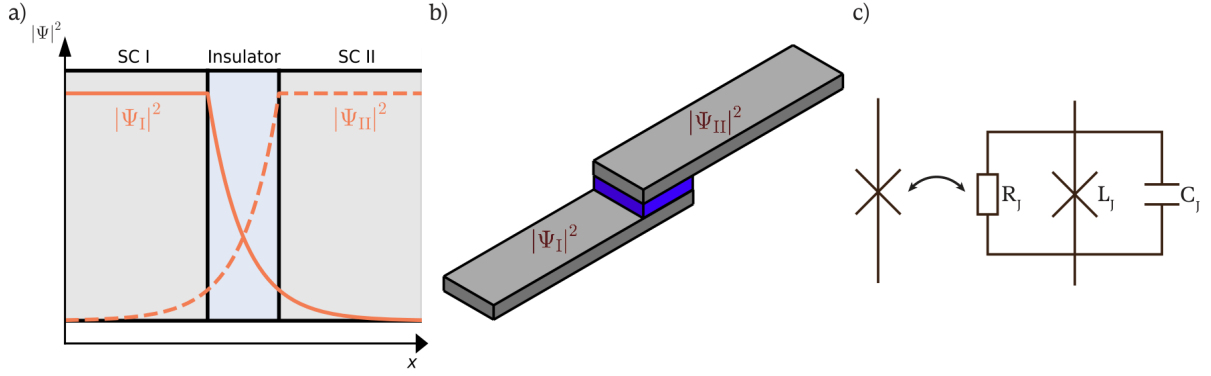


Fig. 2.4 (a) Schematic representation of a Josephson junction showing the overlap of the macroscopic wavefunction Ψ . (b) Illustration of a Josephson junction, where two superconducting electrodes are separated by an insulating barrier. (c) Circuit representation of an ideal Josephson junction and its complete depiction within the RCSJ model, including junction resistance R_J , capacitance C_J , and inductance L_J . Figure and caption reproduced from Ref. [16].

and interact, as illustrated schematically in Fig. 2.4. This coupling allows Cooper pairs to tunnel through the barrier without dissipation [17, 18]. The current and voltage across the barrier depend on the phase difference

$$\phi = \phi_L - \phi_R$$

between the superconducting wavefunctions on each side. The voltage drop V_J and the supercurrent I_J are described by the Josephson relations:

$$I_J = I_0 \sin(\phi), \quad (2.18)$$

$$V_J = \frac{\hbar}{2e} \frac{d\phi}{dt}. \quad (2.19)$$

Here, I_0 is the critical current, the maximum current through the barrier that does not cause dissipation. Its value depends on the junction geometry, the barrier material, and the temperature. The phase difference ϕ discussed in this section so far is the superconducting phase difference between the two sides of the Josephson junction. Equation 2.19, however, allows us to directly map the superconducting phase difference to the generalized flux by

$$\Phi = \frac{\hbar}{2e} \phi = \frac{\Phi_0}{2\pi} \phi. \quad (2.20)$$

Combining the time derivative of Eq. 2.18 with the voltage relation in Eq. 2.19 yields the voltage–current relation

$$V_J = \frac{\hbar}{2e} \frac{1}{\cos \phi} \frac{dI_J}{dt} = \frac{L_{J,0}}{\cos \phi} \frac{dI_J}{dt}. \quad (2.21)$$

Comparing this to the inductive relation in Eq. 2.5, we see that a Josephson junction behaves as a non-linear inductance. The linearised junction inductance is

$$L_{J,0} = \frac{\hbar}{2e} \frac{1}{I_0} = \frac{\Phi_0}{2\pi} \frac{1}{I_0}. \quad (2.22)$$

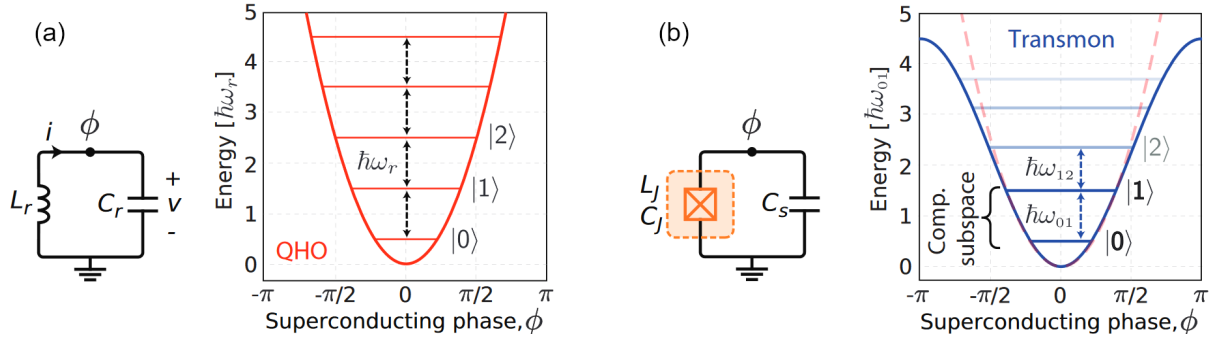


Fig. 2.5 Harmonic and anharmonic oscillator. (a) Circuit model of an LC oscillator. The linear elements result in a quadratic energy potential and equidistant energy levels. (b) The same circuit with a Josephson junction replacing the linear inductance. The nonlinear element modifies the energy potential to a cosine shape and shifts the higher energy levels downward. Figure taken from [19].

For small phase differences ϕ , a Josephson junction can therefore be approximated as a linear inductor. As indicated in Fig. 2.4(c), a real Josephson junction also exhibits capacitive and resistive behaviour. The capacitance arises from the conductor–insulator–conductor geometry of the junction. In many practical cases it is sufficiently small to be neglected, but in Chapter 3 we will see that it provides a significant contribution to the first prototype. The resistive part becomes relevant only when the current exceeds the critical current; since this is outside the regime of interest for our work, we will not further discuss this characteristic.

A third quantity commonly used to characterise a Josephson junction is the Josephson energy E_J , which is proportional to the critical current and expressed in units of energy

$$E_J = \frac{\hbar}{2e} I_0 = \left(\frac{\hbar}{2e} \right)^2 \frac{1}{L_{J,0}}. \quad (2.23)$$

2.2.3 Quantum Mechanical Oscillator

In Sec. 2.1.3 we briefly introduced the harmonic oscillator using an LC circuit. In this section, we examine the same system quantum mechanically and subsequently compare its behaviour to the case where the inductance is provided by a Josephson junction instead of a linear inductor.

Linear Inductance

Let us first consider the simple LC circuit shown in Fig. 2.5(a). The Hamiltonian (Eq. 2.7) of the system becomes an operator when the classical variables are replaced by the corresponding quantum operators

$$\Phi \rightarrow \hat{\Phi}, \quad Q \rightarrow \hat{Q}, \quad H \rightarrow \hat{H}. \quad (2.24)$$

The commutation relation between the two conjugate operators is

$$[\hat{\Phi}, \hat{Q}] = \hat{\Phi}\hat{Q} - \hat{Q}\hat{\Phi} = i\hbar. \quad (2.25)$$

To simplify the Hamiltonian, we introduce the creation and annihilation operators via

$$\hat{\Phi} = \sqrt{\frac{\hbar Z}{2}} (\hat{a} + \hat{a}^\dagger), \quad \hat{Q} = -i\sqrt{\frac{\hbar}{2Z}} (\hat{a} - \hat{a}^\dagger) \quad \text{with} \quad Z = \sqrt{\frac{L}{C}}. \quad (2.26)$$

Using these definitions and the commutation relations, the Hamiltonian can be rewritten in the standard harmonic-oscillator form

$$\hat{H} = \frac{\hat{\Phi}^2}{2L} + \frac{\hat{Q}^2}{2C} = \hbar\omega_r \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad (2.27)$$

where $\omega_r = 1/\sqrt{LC}$. The operator $\hat{a}^\dagger \hat{a}$ corresponds to the photon number operator $\hat{n}$. In its eigenbasis, we have $\hat{n} |n\rangle = n |n\rangle$, giving us the photon number n of a certain state. As depicted in Fig. 2.5(a), the energy potential rises quadratically with the phase Φ . The energy steps of the system are equidistant and the energy increases with the number of photons in the system.

When energy is added to the system by a resonant drive tone, a coherent state is generated [20]. Even for drives in the single-photon regime, such a state populates not only the first excited state but also higher energy levels, with decreasing probability for increasing level index. Therefore, such a resonator cannot be directly used as a qubit, which requires two energy states with an addressable transition and without leakage to higher levels.

Non-linear Inductance

Replacing the linear inductor with a Josephson junction fundamentally changes the system behaviour. This forms the basis of the transmon qubit. The potential in Φ changes from a quadratic form to a cosine

$$V_{\text{lin}}(\Phi) = \frac{\Phi^2}{2L} \quad \rightarrow \quad V_J(\Phi) = E_J \cos \left(\Phi \frac{2\pi}{\Phi_0} \right), \quad (2.28)$$

as shown in Fig. 2.5(b). The height of the cosine potential is set by the Josephson energy E_J of the junction. We can remove the unit for Φ in this equation by introducing the reduced flux $\phi = 2\pi\Phi/\Phi_0$, which is identical to the superconducting phase in equation 2.20. For small phase excursions (small ϕ), the cosine can be expanded in a Taylor series

$$\hat{H} = \frac{\hat{Q}^2}{2C} - E_J \cos \hat{\phi} = -E_J + \underbrace{\left(\frac{\hat{Q}^2}{2C} + \frac{E_J}{2} \hat{\phi}^2 \right)}_{\text{Harmonic oscillator}} - \underbrace{\left(\frac{E_J}{24} \hat{\phi}^4 + \mathcal{O}(\hat{\phi}^6) \right)}_{\text{Higher-order terms}}. \quad (2.29)$$

The cosine potential reduces the energy difference for the higher energy levels, breaking the equidistant level spacing, as illustrated in Fig. 2.5(b). The leading non-harmonic contribution is the quartic term $\propto \hat{\phi}^4$. Keeping only energy-conserving terms of this non-linearity yields the Kerr Hamiltonian

$$\hat{H} = \hbar\omega_r \hat{a}^\dagger \hat{a} - \hbar \frac{\mathcal{K}}{2} \hat{a}^\dagger \hat{a}^\dagger \hat{a} \hat{a}, \quad (2.30)$$

where $\mathcal{K} > 0$ is the Kerr constant. The energy of the Fock state $|n_c\rangle$ becomes

$$\langle n_c | \hat{H} | n_c \rangle = n_c \hbar\omega_r - (n_c - 1) \hbar \mathcal{K}. \quad (2.31)$$

Thus, each additional photon shifts the transition frequency to the next level by $\mathcal{K}$. For the transmon, where the Josephson junction provides the entire inductance, the anharmonicity is given by $\mathcal{K} = E_C/\hbar = e^2/(2\hbar C)$ [19]. If the transition linewidths are much smaller than this anharmonicity, an incoming probe tone at an appropriate frequency will drive only the $|0\rangle \rightarrow |1\rangle$ transition, and the system behaves as an effective two-level system—even though higher levels still exist and influence the dynamics.

So far, we have considered the limiting cases of a purely linear inductor or a purely Josephson inductance. In larger circuits, several resonant modes exist, and each mode typically has contributions from both types of inductances. The resulting potential is a mixture of quadratic and cosine terms. Regardless of the exact form of the potential, the quadratic part always describes the linear response, making Eq. 2.30 a good approximation. The anharmonicity of such a circuit can be calculated by

$$\mathcal{K} = \frac{\eta E_C}{\hbar} = \frac{\eta e^2}{2\hbar C}, \quad (2.32)$$

with η being the participation ratio of the junction and C the mode capacitance [21]. The smaller the Josephson participation η , the smaller the Kerr non-linearity $\mathcal{K}$ becomes. The participation ratio of the junction as well as the mode capacitance vary from mode to mode and are not trivial to calculate in a distributed circuit.

2.3 SQUIDS

A superconducting loop with two Josephson junctions is called a superconducting quantum interference device (SQUID), as illustrated in Fig. 2.6. SQUIDS are widely used as extremely sensitive magnetic-field sensors. We will now derive the behaviour of a SQUID loop in an applied magnetic field. At the beginning we will assume, for simplification, that the superconducting connections do not have any inductance.

2.3.1 SQUIDS without geometrical inductance

The junctions on both sides are generally not identical and have different critical currents $I_0^{(1)}$ and $I_0^{(2)}$. It is useful to introduce an asymmetry parameter α_I and the mean critical current I_0^{JJ} , defined as

$$\alpha_I = \frac{|I_0^{(1)} - I_0^{(2)}|}{I_0^{(1)} + I_0^{(2)}}, \quad (2.33)$$

$$I_0^{\text{JJ}} = \frac{I_0^{(1)} + I_0^{(2)}}{2}. \quad (2.34)$$

Note that we use I_0^{JJ} for the mean critical current of the two SQUID junctions, in order to distinguish it well from the critical current I_0 of a single junction. An applied current I across the SQUID loop is divided into the two branches and can be expressed as a function of the phase drops across each junction, ϕ_1 and ϕ_2 , as

$$I = I_1 + I_2 \quad (2.35)$$

$$= I_0^{(1)} \sin(\phi_1) + I_0^{(2)} \sin(\phi_2) \quad (2.36)$$

$$= I_0^{\text{JJ}}(1 - \alpha_I) \sin(\phi_1) + I_0^{\text{JJ}}(1 + \alpha_I) \sin(\phi_2) \quad (2.37)$$

$$= 2I_0^{\text{JJ}}(\sin(\phi_1) + \sin(\phi_2)) + 2\alpha_I I_0^{\text{JJ}}(\sin(\phi_2) - \sin(\phi_1)) \quad (2.38)$$

$$= 2I_0^{\text{JJ}} \sin\left(\frac{\phi_1 - \phi_2}{2}\right) \cos\left(\frac{\phi_1 + \phi_2}{2}\right) + 2\alpha_I I_0^{\text{JJ}} \cos\left(\frac{\phi_1 - \phi_2}{2}\right) \sin\left(\frac{\phi_1 + \phi_2}{2}\right), \quad (2.39)$$

To remove a degree of freedom and to include the externally applied flux, we have to consider the quantisation condition inside a closed superconducting loop, the so-called *fluxoid quantisation*

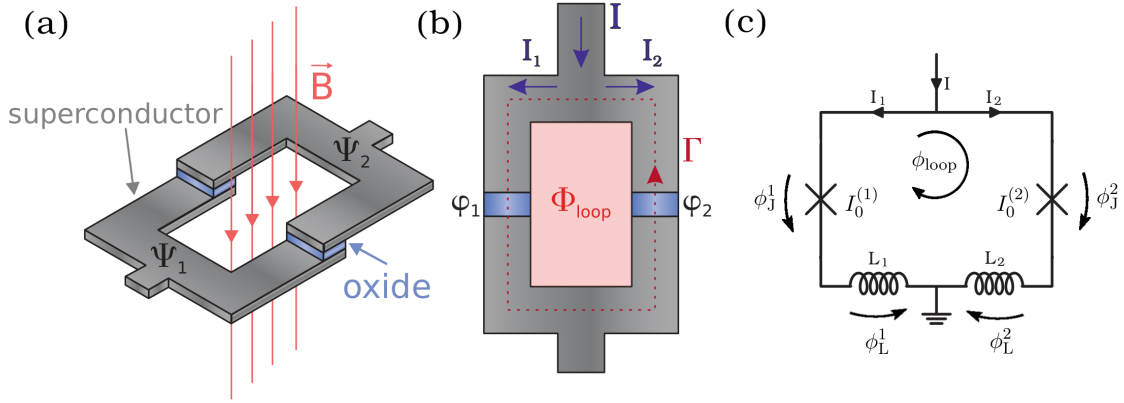


Fig. 2.6 Illustration of a SQUID. A superconducting loop is interrupted by two Josephson junctions. Externally applied flux leads to circulating currents and frustration of the junctions. Picture taken from [22].

condition. It states that the externally applied flux through the loop Φ_{loop} and the algebraic sum of all branch fluxes inside the loop must sum to an integer multiple of the flux quantum. To show this, one performs a closed-loop integral over the phase of the superconducting wavefunction as done in section 2.2.1. In our case this leads to

$$2\pi \left(n - \frac{\Phi_{\text{loop}}}{\Phi_0} \right) = \phi_2 - \phi_1. \quad (2.40)$$

As we now consider the case of zero linear loop inductance, we only have to account for the phase drop across the junctions. Furthermore, the loop flux Φ_{loop} is identical to the externally applied flux Φ_{ext} . Inserting this into Eq. 2.39 we obtain

$$\begin{aligned} I &= 2I_0^{\text{JJ}} \sin\left(\frac{\phi_1 - \phi_2}{2}\right) \cos\left(\frac{\phi_1 + \phi_2}{2}\right) + 2\alpha_I I_0^{\text{JJ}} \cos\left(\frac{\phi_1 - \phi_2}{2}\right) \sin\left(\frac{\phi_1 + \phi_2}{2}\right) \\ &= 2I_0^{\text{JJ}} \sin\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right) \cos\left(\frac{\phi_1 + \phi_2}{2}\right) + 2\alpha_I I_0^{\text{JJ}} \cos\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right) \sin\left(\frac{\phi_1 + \phi_2}{2}\right) \\ &= 2I_0^{\text{JJ}} \cos\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right) \left[\sin\left(\frac{\phi_1 + \phi_2}{2}\right) + \alpha_I \tan\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right) \cos\left(\frac{\phi_1 + \phi_2}{2}\right) \right] \\ &= 2I_0^{\text{JJ}} \cos\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right) \sqrt{1 + \alpha_I^2 \tan^2\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right)} \sin\left(\frac{\phi_1 + \phi_2}{2} + \arctan\left[\alpha_I \tan\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right)\right]\right) \\ &= 2I_0^{\text{JJ}} \cos\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right) \sqrt{1 + \alpha_I^2 \tan^2\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right)} \sin(\phi'), \end{aligned}$$

where in the second-to-last line we used the trigonometric identity

$A \sin(\omega) + B \cos(\omega) = \sqrt{A^2 + B^2} \sin(\omega + \arctan(B/A))$. In the last line we introduced a new effective phase ϕ' . By comparing the resulting equation with Eq. 2.18 we see that the critical current of a SQUID behaves similarly to that of a single junction. The difference is that the effective critical SQUID current

$$I_0^{\text{SQUID}} = 2I_0^{\text{JJ}} \cos\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right) \sqrt{1 + \alpha_I^2 \tan^2\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right)} \quad (2.41)$$

depends on the junction parameters and on the effective flux through the SQUID loop, Φ_{loop} . It is a periodic function of the applied magnetic field with period Φ_0 . For symmetric junctions,

the critical current goes to zero when $\Phi_{\text{loop}} = \Phi_0/2$, while for asymmetric junctions it remains finite. Analogous to Eq. 2.22, we can introduce the inductance of a SQUID loop as a function of magnetic flux:

$$L_{\text{SQUID}}(\Phi_{\text{loop}}) = \frac{\hbar}{2e} \frac{1}{I_0^{\text{SQUID}}(\Phi_{\text{loop}})} = \frac{\hbar}{4eI_0^{\text{JJ}} \cos\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right) \sqrt{1 + \alpha_I^2 \tan^2\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right)}}. \quad (2.42)$$

It should be emphasised that this expression is only valid when any linear inductance in the SQUID loop is neglected.

2.3.2 SQUIDS with geometrical inductance

The superconducting leads of a SQUID loop naturally carry linear inductance. In a system with linear loop inductance, circulating currents within the loop generate magnetic fields that either counteract or reinforce the external field. For sufficiently large linear inductances this can lead to multiple stable solutions with different circulating-current directions and strengths. A simple analytical expression for the critical current or inductance of the SQUID loop is therefore no longer possible. If the junction inductance is much larger than the linear inductance, the system can be well approximated by neglecting the influence of the linear inductance.

However, the smaller the cross-section of the leads and the longer the circumference of the loop, the larger the linear inductance becomes. In the case of our design, neglecting this linear inductance is not possible. We will not discuss the most general case, as done for example in [22], but instead focus on the case of zero externally applied DC-bias current across the SQUID loop. This is sufficient in our situation, as the SQUID is neither part of a closed loop where external fields generate DC currents, nor connected to external DC sources [23]. A useful dimensionless parameter to describe the ratio between linear and Josephson inductance is the screening parameter

$$\beta_L = \frac{2LI_0^{\text{JJ}}}{\Phi_0} = \frac{L}{\pi L_0^{\text{JJ}}}, \quad (2.43)$$

with the total linear inductance $L = L_1 + L_2 - 2M$ composed of the inductances of the left and right branches as well as the mutual inductance M between them. The junction inductance can be expressed in terms of the mean critical current I_0^{JJ} (see Eq. 2.34) or through the corresponding inductance $L_0^{\text{JJ}} = \Phi_0/(\pi I_0^{\text{JJ}})$. There are different ways to analyse this problem; for example, Christian Schneider uses loop currents to relate Φ_{loop} and Φ_{ext} [22]. In this work, however, we will follow a more direct approach by working with the phases.

In the general case, the quantisation condition from Eq. 2.40 must also include the phase drop across the linear inductors as indicated in Fig. 2.6(c). This leads to the equation

$$2\pi \left(n - \frac{\Phi_{\text{loop}}}{\Phi_0} \right) = \phi_J^2 + \phi_L^2 - \phi_J^1 - \phi_L^1, \quad (2.44)$$

where the indices reference the side of the loop, and L and J refer to the linear inductance and the junction, respectively. The phases on one side of the loop are not independent because they experience the same current. Setting the junction and inductance currents equal and using

Eq. 2.18 for the Josephson current gives the condition

$$\phi_L^i = \frac{L_i}{L_{J,0}^i} \sin(\phi_J^i). \quad (2.45)$$

Applying this to both branches reduces the number of free parameters in Eq. 2.44 to two:

$$2\pi \left(n - \frac{\Phi_{\text{loop}}}{\Phi_0} \right) = \phi_J^2 + \frac{L_2}{L_{J,0}^2} \sin(\phi_J^2) - \phi_J^1 - \frac{L_1}{L_{J,0}^1} \sin(\phi_J^1). \quad (2.46)$$

A physical $\phi_J^1 - \phi_J^2$ configuration must satisfy this equation. The phase configuration in operation can be found by selecting the combination that minimises the total inductive energy of the SQUID. An easy way to find this configuration is to compute the inductive energy of the allowed phase configurations and pick the one with the smallest inductive energy, as visualised in Fig. 2.7. There are several ways to rewrite this equation and solve it. In our case, we rewrite Eq. 2.46 as

$$B = \phi_J^1 + A \cdot \sin(\phi_J^1), \quad (2.47)$$

$$A = \frac{L_2}{L_{J,0}^2}, \quad (2.48)$$

$$B = 2\pi \left(n - \frac{\Phi_{\text{loop}}}{\Phi_0} \right) + \phi_J^1 + \frac{L_1}{L_{J,0}^1} \sin(\phi_J^1), \quad (2.49)$$

and calculate ϕ_J^2 on a fine grid of $\phi_J^1 \in [-\pi, \pi]$. For small linear inductances $L_2/L_{J,0}^2 < 1$, there is only one stable ϕ_2 for every ϕ_1 and vice versa. In general, however, two or more stable phase configurations can occur, as shown in Fig. 2.7.

Multiple local minima can appear along the allowed ϕ_1, ϕ_2 curves. When the magnetic field is tuned, the curves change shape and position, and the local minima shift accordingly. This leads to hysteresis: the operational state will not always be the energetically most favourable one but can remain trapped in a local minimum when the field is tuned slowly. A jump to another minimum occurs only when thermal or other fluctuations overcome the barrier separating the two minima, or when the barrier collapses. This typically happens when the loop current and thus also the junction inductances are high. At some point, the state will jump to a more favourable configuration (if such a state exists). This hysteretic behaviour occurs only for a screening parameter $\beta_L > 1$. In Fig. 2.8 it is shown how the inductance of a SQUID tunes for different parameters. In Fig. 2.8, only the lowest-energy state is shown, and hysteretic effects are therefore not visible. An example of such hysteretic behaviour is presented in Ref. [22].

In this chapter we discussed the fundamentals of electrical circuits, analysed LC oscillators from a quantum-mechanical perspective, and examined the behaviour of a SQUID under an in-plane applied magnetic field. From this point onward, we consider more complex circuits containing multiple SQUID loops, inductances, and capacitances. With the foundations discussed so far, it should now be possible to understand their behaviour and characteristic properties.

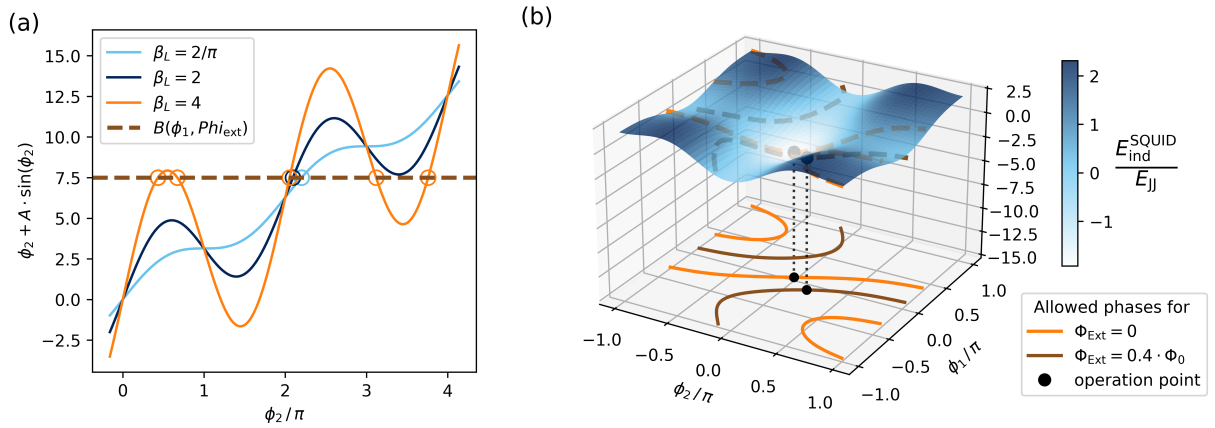


Fig. 2.7 SQUID phases and energy. (a) Equation 2.49 visualised. The equation is fulfilled when the right side (sinusoidal curves) equals the dotted horizontal line (B), indicated by small circles. The amplitude of the sine increases with higher geometrical inductance, leading to several possible ϕ_2 values for one ϕ_1 . (b) All possible ϕ_1 - ϕ_2 combinations for a double-SQUID system with $\beta_L = 0.9$ and $\alpha_I = 0.15$ at two different external magnetic-flux values. Changing the magnetic field changes the shape of the curves adiabatically. On the third axis the inductive energy of the SQUID loop is shown. Along the allowed ϕ_1, ϕ_2 curves, several local minima or maxima can appear. The operational state is not necessarily the global minimum.

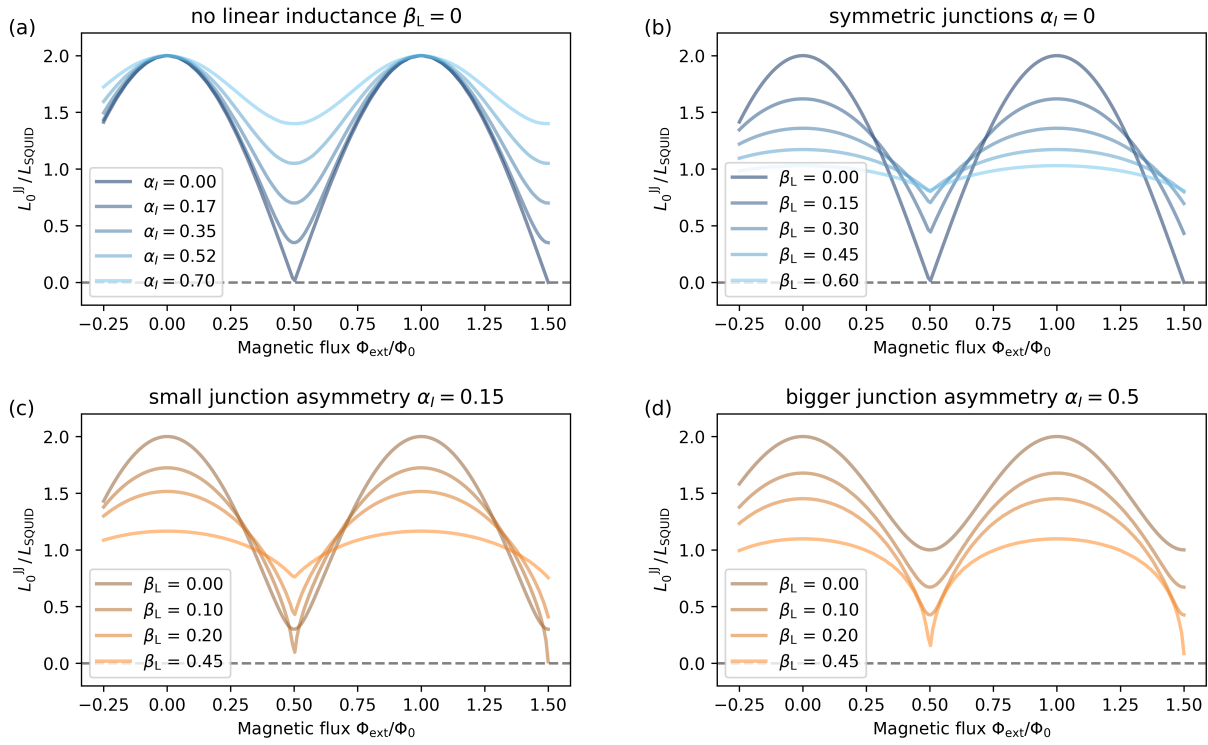


Fig. 2.8 SQUID inductance for different properties, the geometry of the SQUID is assumed to be symmetric $L_1 = L_2$. The plot only shows the energetically most favourable state. (a): System with no lead inductance, the SQUID inductance in this case can be calculated analytically with equation 2.42. (b) Identical junctions in both leads. (c, d) Combination of junction asymmetry and loop inductance.

CHAPTER 3

Absolute field sensing: Double SQUID circuit

A single SQUID loop used as a magnetic-field sensor faces a fundamental limitation: achieving high sensitivity requires a large loop area. The SQUID response to an applied magnetic field is inherently periodic. For loop sizes that provide sufficient sensitivity, this period is typically much smaller than the offset field to be measured.

For example, a SQUID with a loop area of $10\,000\,\mu\text{m}^2$ has a magnetic-field period of approximately $0.2\,\mu\text{T}$. Compared to the Earth's magnetic field of about $50\,\mu\text{T}$, this corresponds to roughly 250 periods. As a consequence, without additional information from other magnetic-field sensors, it is not possible to determine the absolute magnetic-field offset from a single SQUID measurement alone.

In this chapter, we present one possible approach to overcome this limitation. The central idea of this project is to combine multiple SQUID loops within a single device. The combination of several periodic responses gives rise to an effective periodicity that is significantly larger than that of an individual SQUID loop.

We begin by discussing the general operating principle of a system consisting of two SQUID loops and by analysing the fundamental limitations of this approach. Subsequently, we introduce a specific implementation of a double-SQUID system based on transmon-like antenna pads. Finally, we examine the two fabricated prototypes and motivate the corresponding design choices based on the behaviour of the general system.

3.1 Loop area ratio and absolute field measurement

To begin this chapter, we first consider the general problem: how do two periodic signals behave when they are combined? For now, we will not consider details such as how the signals are combined; instead, we assume that we are either able to measure them simultaneously or that they interact in some way such that we can measure a mixed response.

To simplify the vocabulary, we consider periodic signals in time, keeping in mind that time can be replaced by any other variable, such as magnetic field. Each signal i has a certain period T_i after which it repeats, and an associated frequency $\omega_i = 2\pi/T_i$.

To investigate what happens when signals with different periods T_i are combined, we first consider the special case $T_1 = 2 \cdot T_2$. This example is illustrated in Fig. 3.1 in light blue, showing a cosine and a sine signal. The period of the combined signal is simply $T_{\text{comb}} = T_1$, since signal 2 repeats itself exactly twice within this period. The more interesting case arises when the periods differ by a non-integer factor, for example $T_1 = 2.1 \cdot T_2$, shown in Fig. 3.1 in dark blue. The new total period can be found by identifying the smallest integer multiple of T_1 that is also an integer multiple of T_2 . In this case, the period of the combined signal is $T_{\text{comb}} = 10 \cdot T_1 = 21 \cdot T_2$. We will investigate the limitations of this viewpoint in the next section. For now, we apply this concept to our system: the combination of two SQUID loops.

Each loop has an inductance that changes when a magnetic field is applied, or more specifically, when magnetic flux penetrates the loop. This inductance can be measured by determining the resonance frequency of a LC-circuit in which the SQUID is embedded. To simplify the considerations in this chapter, we restrict ourselves to SQUID loops without linear inductance. In this case, the response is non-hysteretic, and shielding currents do not need to be considered.

The superconducting phase differences across the junctions adjust such that the flux quantisation condition is always fulfilled. The measured signal (inductance or resonance frequency) therefore has a periodicity of one flux quantum Φ_0 . More precisely, we know how much of a flux quantum is missing to reach the nearest integer value. Mathematically, this can be expressed as

$$\Delta_{\Phi,i} = \frac{\Phi_{\text{ext}}^i}{\Phi_0} \bmod 1, \quad (3.1)$$

where i indexes the loop. While $\Delta_{\Phi,i}$ can be measured, it is not possible to determine the total flux Φ_{ext}^i through a single loop. Introducing a second independent loop allows us to overcome this limitation, provided its loop size is chosen appropriately.

For an applied magnetic field B_{ext} , the flux through a loop depends on its area A_i

$$\Phi_{\text{ext}}^i = B_{\text{ext}} A_i. \quad (3.2)$$

The case of inhomogeneous fields will be discussed later in Section 3.1.2. For now, we assume a homogeneous magnetic field with $B_{\text{ext}}^1 = B_{\text{ext}}^2$. In this case, the magnetic fluxes through the two loops are proportional to each other and determined by the geometrical area ratio

$$\frac{\Phi_{\text{ext}}^1}{\Phi_{\text{ext}}^2} = \frac{A_1}{A_2} = \frac{n}{m} = k. \quad (3.3)$$

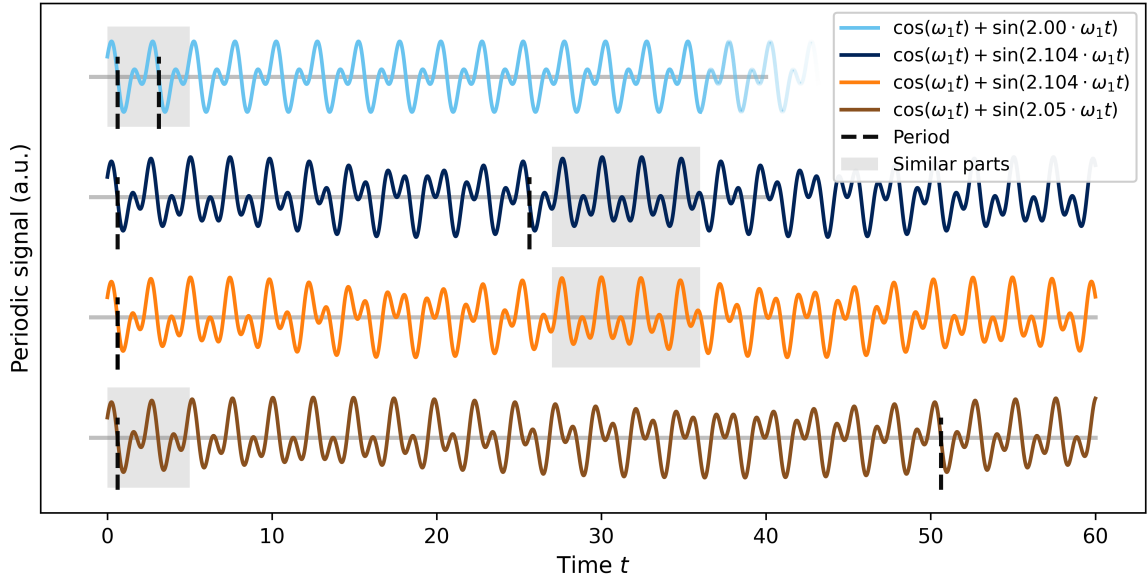


Fig. 3.1 Illustration of the increased period obtained by combining two periodic signals. The graphs show the sum of a cosine and a sine function with different frequencies. In light blue, both frequencies are integer multiples of each other, resulting in a short overall period. The lower graphs show slightly mismatched frequencies, which significantly increase the overall period. For frequencies close to an integer ratio, it can be difficult to identify the point of exact phase match.

In principle, any loop ratio k can be used. For clarity, we restrict ourselves to integer coprime numbers n and m with $n > m$. If we plot each loop response as a function of the applied magnetic field, each signal has a period

$$T_i^B = \frac{\Phi_0}{A_i}, \quad (3.4)$$

where the superscript B indicates that T^B is a magnetic-field interval rather than a time period. The total periodicity of the combined system is then

$$T_{\text{comb}}^B = T_1^B \cdot n = T_2^B \cdot m. \quad (3.5)$$

Knowing the combined periodicity alone is not sufficient for field measurement. Crucially, we can measure $\Delta_{\Phi,i}$ for both loops. The reference point of interest is the zero-field condition $B_{\text{ext}} = 0$, for which

$$\Delta_{\Phi,1} = \Delta_{\Phi,2} = 0. \quad (3.6)$$

We refer to Eq. 3.6 as the *phase-match condition*. At zero magnetic field, this condition is fulfilled, but it is not the only point where this occurs. As the magnetic field increases, Φ_{ext}^i evolves according to Eq. 3.2, and consequently $\Delta_{\Phi,i}$ changes. While $\Delta_{\Phi,1}$ and $\Delta_{\Phi,2}$ vary, their relative evolution is fixed by the area ratio. The phase-match condition is not fulfilled again until the combined period T_{comb}^B is reached.

We can summarise the conclusions as follows:

- To use a double-SQUID system as an absolute magnetic-field sensor, the external magnetic field must be tunable continuously, using e.g. an additional coil. Measuring the response over a field sweep yields a periodic signal.
- The period $T_{\text{comb}}^{\text{B}}$ of this response can be much larger than that of a single SQUID loop, but it remains periodic. Within each period $T_{\text{comb}}^{\text{B}}$, there exists a point where the phase-match condition of Eq. 3.6 is fulfilled.
- The exact appearance of this point depends on the interaction between the two SQUID-loop signals, but it is a unique feature that can be predicted theoretically.
- Because the signal is periodic, the phase-match condition is fulfilled at several points, separated by $T_{\text{comb}}^{\text{B}}$, all of which are possible offset fields. From the spectrum alone, these points are indistinguishable.
- To identify the true offset field, an approximate prior knowledge of the expected field is required. The areas of the device have to be chosen in a way that $T_{\text{comb}}^{\text{B}}/2 > \text{uncertainty of the prior knowledge}$.

In the following section, we discuss the limitations of the simplified picture presented here.

3.1.1 Limitations

Looking at the mathematics above, it appears that one could choose arbitrarily large values for n and m . This would lead to very large combined periods and, naively, to negligible restrictions on how accurately the offset field must be known. However, there are several important limitations to consider.

One practical limitation is fabrication imperfections. It is not possible to fabricate an exact area ratio; there will always be deviations. In order to achieve good reproducibility, variations in loop size due to fabrication tolerances should not significantly change the behaviour of the system.

Another important consideration is how small deviations in the area ratio affect the measured spectrum. This is closely connected to the restriction imposed earlier on the area ratio $k = n/m$. To illustrate this, let us compare a device A with an area ratio $k_{\text{A}} = 21/10 = 2.1$ to a device B with $k_{\text{B}} = 21.04/10 = 2.104$. The first device has a combined periodicity of 21 times the big-loop period, while the second device, in principle, has a period of 263 times the big-loop period. Physically, however, both systems are nearly identical, and their signals will appear very similar.

The signal of a periodic function with frequency (area) ratio of $k_{\text{A}} = 2.1$ is plotted in Fig. 3.1 (dark blue). Underneath we plotted the a similar periodic function with $k_{\text{B}} = 2.104$ in orange, the visible response is almost indistinguishable within the shown range. Although system B formally exhibits a much larger period, it also contains a much smaller sub-period, equal to that of system A. For example, at 21 times the small-loop period one finds $\Delta_1 = 21.04 \bmod 1 = 0.04$ and $\Delta_2 = 10 \bmod 1 = 0$. Strictly speaking, this does not satisfy the phase-match condition; practically, however, this point may not be distinguishable from one where the phase-match condition is fulfilled.

The same argument applies if the restriction $k = n/m$ is relaxed and irrational area ratios are allowed, which is the more realistic case when considering fabrication imperfections. While the periodicity is formally infinite, in practice it is always possible to approximate the area ratio by a rational number $k = n/m$. In addition, smaller values of n and m may exist that produce a spectrum very similar to the measured one, but with a significantly smaller effective period. Taking the example from above again, the orange trace with $k = 2.104 = 263/125$ could be approximated by a signal with an area ratio $k = 2.1 = 21/10$. The same could be done for any irrational value between these two numbers.

A special case arises when the area ratio approaches an integer value. As indicated in Fig. 3.1 (brown), a ratio of $k = 2.05$ already yields a substantially larger period than the case $k = 2.10$ (dark blue). The trade-off for the increased period, however, is that when only a small portion of the fluxmap is considered (e.g. the interval $[0, 5]$ in Fig. 3.1), the signal appears nearly periodic, similar to the light blue curve corresponding to $k = 2$, as indicated with the gray box. This effect will play a crucial role later when discussing the influence of inhomogeneous magnetic fields.

In Fig. 3.2(a), the combined period $T_{\text{comb}}^{\text{B}}$ of two SQUID loops is plotted as a function of the selected loop area ratio. To simplify the discussion, we move away from the exact rational representation $k = n/m$ and instead consider ratios close to an integer value. In this regime, the ratio can be written as $k = k_{\text{int}} \pm k_{\delta}$, where k_{int} is an integer and $k_{\delta} \ll 1$ denotes a small deviation.

It should be noted that the area ratios shown in Fig. 3.2 are calculated using the smallest possible integer values of m and n . In a real device, the actual area ratio will be a rational number close to the intended one due to fabrication uncertainties. In that case, the values of $T_{\text{comb}}^{\text{B}}$ shown here represent the smallest sub-period of the resulting spectrum.

Throughout Chapters 3 and 4, we deliberately neglect the possibility of larger effective periods in order to obtain conservative bounds on the achievable period and gradient sensitivity. In practical measurements, however, this effect can relax the period constraint and thereby improve the usable operating range of the device. This can especially occur for ratios far from an integer value ($k_{\delta} > 0.05$) in combination with high measurement resolution.

3.1.2 Effects of field inhomogeneity

As motivated in the previous chapter, the idea of absolute field measurement is based on a “phase” comparison between the signals of two SQUID loops. The phase-match condition in Eq. 3.6 relies on the fact that the magnetic flux through each loop is zero (or an integral multiple of Φ_0). Inhomogeneous magnetic fields influence this “phase”, as they shift the responses of each of the SQUID loops independently. There are several points that nearly fulfil the phase-match condition, and a sufficiently large shift due to an inhomogeneous field can cause one of these points to satisfy it instead. In this section, we investigate how this affects the measurement of the offset field.

First, two different types of gradients must be distinguished: gradients in the offset field and gradients in the tuning field used to change the magnetic field and record the behaviour/signal of each SQUID loop. To determine the offset field, one aims to find the magnetic field at which

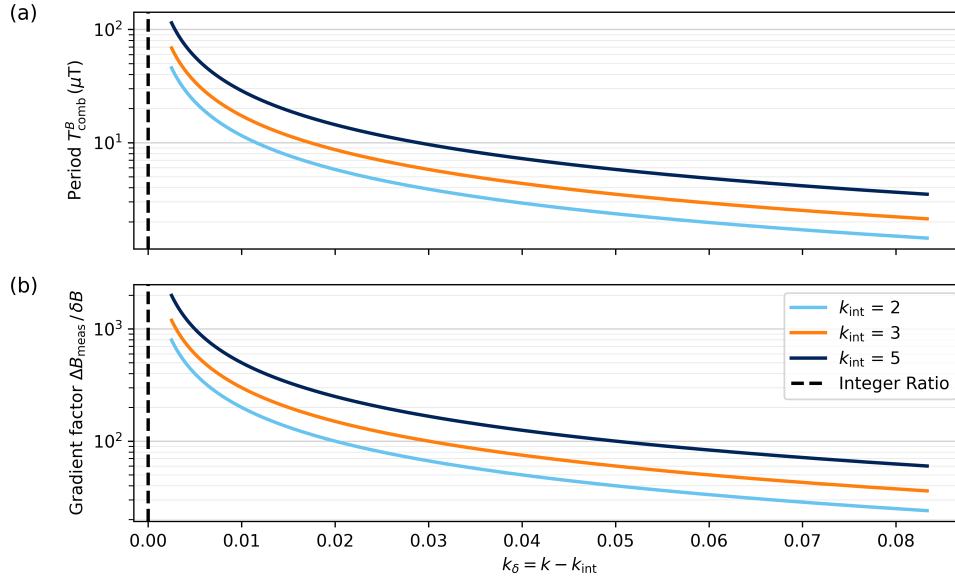


Fig. 3.2 Combined period and gradient sensitivity as a function of the loop area ratio. The area ratio is defined as $k = k_{\text{int}} + k_{\delta} = n/m$. Integer values of n and m were chosen by the restriction $k_{\delta} = 1/m$, which ensures the smallest possible choice of n and m for a certain k . (a) Combined period of both SQUID loops, calculated using Eq. 3.5. The size of the bigger SQUID loop is fixed to $190 \mu\text{m} \times 190 \mu\text{m}$. (b) Gradient sensitivity, calculated using Eq. 3.16. On the y-axis we plot $\Delta B_{\text{meas}}/\delta B$. Here, ΔB_{meas} denotes the measurement error of the offset field induced by the gradient, while δB quantifies the gradient via the effective field difference between the two SQUID loops.

the phase-match between the loops occurs. Typically, this is achieved by sending a current through a coil until the external field is compensated. Even if there is no flux through one loop, the other loop may still experience external flux. This can occur for the following reasons:

- **Homogeneous tuning field and inhomogeneous offset field**

A homogeneous tuning field compensates the magnetic field at both loops in the same way. Any difference in magnetic field between the two SQUID loops that exists without coil current will therefore remain unchanged. In this case, the gradient effect depends only on the offset-field gradient and not directly on the absolute value of the offset field.

- **Inhomogeneous tuning field and homogeneous offset field**

Using an inhomogeneous tuning field to compensate the external field introduces two effects. First, it generates a difference in magnetic field between the two loops at the point where one loop reaches zero flux. In the special case of zero offset field, no tuning field is required to reach the phase-match condition, and no gradient is introduced. In a realistic scenario, however, a tuning coil is required, to generate a magnetic field $B_{\text{ext}}^i(I) \propto I$ at the position of the SQUID loops, where I is the current through the coil. The introduced inhomogeneity $B_{\text{ext}}^1(I) - B_{\text{ext}}^2(I)$ is proportional to the total field generated by the coil. Tuning the field to zero therefore introduces a gradient proportional to the offset field, but independent of the offset-field gradient.

The second effect of an inhomogeneous tuning field is that it modifies the effective area ratio. In practice, one does not measure the geometric area ratio directly, but rather the ratio of magnetic flux penetrating each loop. For a homogeneous tuning field, this is equivalent to the area ratio. For an inhomogeneous tuning field, however, there exists a constant ratio

$$\frac{B_{\text{ext}}^1(I)}{B_{\text{ext}}^2(I)} = R_{\text{grad}}^B, \quad (3.7)$$

between the magnetic-field amplitudes at the two loop positions. Including this in Eq. 3.3 yields

$$\frac{\Phi_{\text{ext}}^1}{\Phi_{\text{ext}}^2} = \frac{B_{\text{ext}}^1(I)A_1}{B_{\text{ext}}^2(I)A_2} = R_{\text{grad}}^B \frac{A_1}{A_2} = R_{\text{grad}}^B k, \quad (3.8)$$

showing that the measured area ratio is modified by the field-inhomogeneity factor R_{grad}^B . Extracting the loop area from the spectrum therefore yields a value that differs from the actual geometric area. Both effects represent different perspectives on the same underlying problem; they are separated here to emphasise the distinction between geometric area ratio and magnetic flux ratio.

- **Combined effects**

In an actual experiment, neither the offset field nor the tuning field will be perfectly homogeneous. Both effects will therefore combine, either partially compensating or reinforcing each other. Experimentally, only the phase-match point between the two signals can be estimated. Without independent knowledge of the field distribution, it is not possible to distinguish whether a shift of the phase-match point is caused by the true offset field or by field-gradient effects.

To quantify the effect of gradients, we consider two spatially separated SQUID loops. Depending on the geometry and the spatial profile of the magnetic field, each loop experiences a magnetic flux $\Phi_{\text{ext},i}$ due to the offset field. For simplicity, we define an effective magnetic field

$$B_i = \frac{\Phi_{\text{ext},i}}{A_i} \quad (3.9)$$

for each loop by dividing the flux by the effective loop area A_i . This quantity can be interpreted as an approximate magnetic offset field at the centroid of the SQUID loop. We now restrict the discussion to area ratios close to an integer value. Let k_{int} be a fixed integer and $k_\delta \ll 1$ a small deviation, such that the absolute area ratio is $k = k_{\text{int}} \pm k_\delta$. In principle, any area ratio can be used, but values close to integers yield spectra that are easier to interpret.

For realistic designs and magnetic fields, the two effective fields will be similar but differ by a small amount

$$\delta B = |B_2 - B_1| \ll \min(T_1^B, T_2^B). \quad (3.10)$$

We now use a coil to reach the point of zero flux in the first loop, $B_1 = 0$. The zero-field condition in Eq. 3.6 is not fulfilled, since the second loop acquires a phase

$$\Delta_{\Phi,2}(\delta B) = \delta B \cdot \frac{A_2}{\Phi_0} \bmod 1 \neq 0. \quad (3.11)$$

This is either due to offset field gradients or due to gradients introduced by the coil. Consequently, the true zero-field point cannot be identified directly. Instead, we must search for the

nearest point where the zero-field condition is fulfilled again. To do so, one would apply a field B_{ext} to reach the next point where

$$\Delta_{\Phi,1} = \Delta B_{\text{ext}} \cdot \frac{A_1}{\Phi_0} \bmod 1 = 0 \quad (3.12)$$

is fulfilled, here the phase on the first SQUID loop is zero. This condition is satisfied when an external field equal to one period of the first loop is applied, $\Delta B_{\text{ext}} = T_1^B$ as defined in Eq. 3.4. Here we assume that the gradient of the sweep coil is small compared to the total generated field, such that $\Delta B_{\text{ext}} \approx B_{\text{ext}}^1 \approx B_{\text{ext}}^2$. From the area ratio, we can determine how much the phase of the second loop changes:

$$\Delta_{\Phi,2}(\Delta_{\Phi,1} = 1) = T_1^B \cdot \frac{A_2}{\Phi_0} \bmod 1 = \frac{\Phi_0}{A_1} \cdot \frac{A_2}{\Phi_0} \bmod 1 = \frac{A_2}{A_1} \bmod 1. \quad (3.13)$$

For arbitrary area ratios, identifying the next point that fulfils the phase-match condition would require to look at $\Delta_{\Phi,1} = 1, 2, 3, \dots$ and find the closest match. Since we restrict ourselves to designs close to integer ratios, we know that the neighboring points are the ones closest to the phase match condition. Thus we only have to look at $\Delta_{\Phi,1} = \pm 1$ which allows us to simplify this to

$$\Delta_{\Phi,2}(\Delta_{\Phi,1} = 1) = \frac{A_2}{A_1} \bmod 1 = (k_{\text{int}} \pm k_{\delta}) \bmod 1 = \pm k_{\delta}. \quad (3.14)$$

Summing up, we know now the phase shift introduced on the second loop because of the offset field. We also know the phase shift on the second loop when we change the magnetic field so that the flux through the first loop is exactly $\Delta_{\Phi,1} = \pm 1$. Now we can ask, how much magnetic field do we have to apply, so that these two phases become the same and the phase match condition thus is fulfilled. The number of flux quanta j_1 that must be added to the first loop to reach the nearest point with $\Delta_{\Phi,2} = \pm 0$ is

$$j_1 = \frac{\Delta_{\Phi,2}(\delta B)}{\Delta_{\Phi,2}(\Delta_{\Phi,1} = 1)} = \frac{\delta B \cdot A_2 / \Phi_0}{\pm k_{\delta}}. \quad (3.15)$$

The corresponding magnetic-field error follows as

$$\Delta B_{\text{meas}} = \frac{j_1 \Phi_0}{A_1} = \delta B \cdot \frac{k}{\pm k_{\delta}}. \quad (3.16)$$

The quantity ΔB_{meas} represents the deviation between the measured offset field and the true offset field. The field estimated from the spectrum, B_{meas} , differs from the actual offset field $B_{\text{off}} \approx B_1 \approx B_2$ by

$$B_{\text{meas}} = B_{\text{off}} \pm \Delta B_{\text{meas}}. \quad (3.17)$$

The correction can have either sign and, depending on the design parameters and the strength of the field gradient, can be comparable in magnitude to the offset field itself. From equation 3.16 we can see, that the influence of the gradient heavily depends on the chosen area ratio. For area ratios with small k_{δ} , which lead to a strong increase in the magnetic field period, the factor k/k_{δ} can become very large. In Fig. 3.2(b), the gradient sensitivity is plotted as a function of the area ratio. The gradient sensitivity scales in the same way as the combined period of the system.

This observation motivates the choice of area ratios that balance an extended periodicity with minimizing sensitivity to gradient-induced measurement errors. Fabrication uncertainties must also be considered when selecting k . In the fabrication process, both loop areas will exhibit deviations of the form $A_i + \delta A_i$. These deviations directly affect the ratio k . As k_δ becomes small, even minor fabrication errors δA_i can induce large variations in k_δ , which in turn strongly influence the resulting period T_{comb}^B .

3.2 Two SQUIDs in series

So far, the discussion has been general and is applicable to systems with two SQUID loops, a magnetic field which can be changed continuously and some way to measure the inductance change of the SQUID loops. Now we will look at the implementation of the concept we have chosen: Both SQUID integrated in a single LC-circuit. In this design, the inductance change of each loop reflects in the modulation of the resonance frequency.

The idea of the chosen design is motivated by the work in [10] and used for the fabricated prototypes. The concept is to build transmon-like antennas and place two SQUIDs in series between the antenna pads. In this configuration, the SQUIDs are not part of a closed loop, so no DC currents can flow through the two SQUID system when a magnetic field is applied. This allows us to treat the SQUIDs independently when investigating their behaviour under an applied magnetic field. Minimizing the energy function of the SQUID subsystem is sufficient to describe the system behaviour in this case [23].

An example of such a system, together with a representative circuit diagram, is shown in Fig. 3.3. The circuit contains Josephson junctions and capacitances represented as lumped elements.

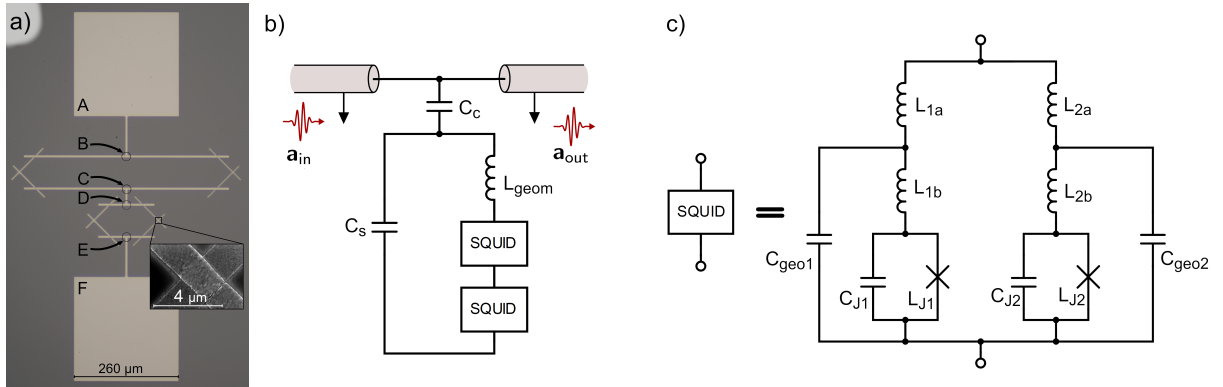


Fig. 3.3 Double SQUID circuit. (a) Picture of the first double-SQUID prototype. The antenna pads (A, F) are interrupted by two SQUID loops with unequal area (B-C and D-E). The electron-beam picture of one junction is shown in detail. (b) Circuit diagram of the device. Geometric inductances and capacitances are distributed and approximated for the calculation using lumped elements between perfect conductors. The antenna pads couple capacitively to the waveguide via C_c . The connection between the SQUIDs introduces an inductance L_{geom} . Furthermore, a shunt capacitance C_s exists between the pads. Within the SQUIDs, there are inductances and capacitances originating both from the geometry and from the junctions.

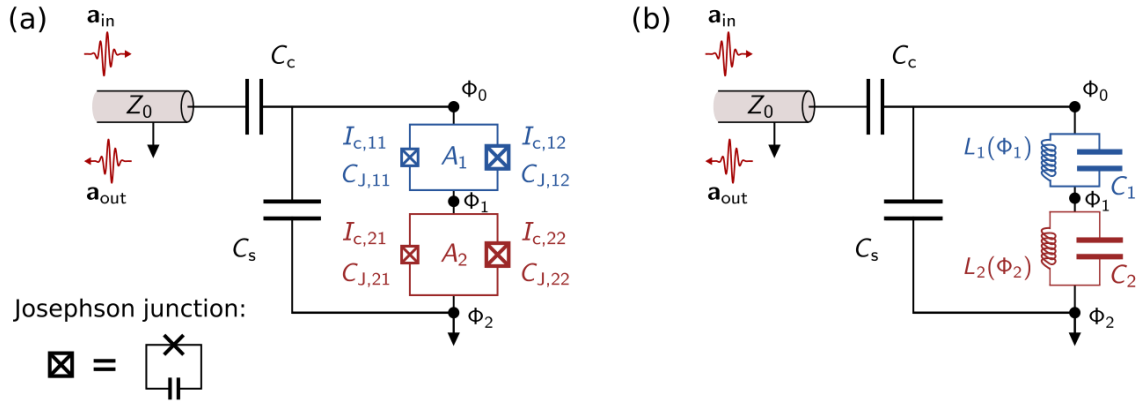


Fig. 3.4 Circuit model without linear inductances. In this limit, it is possible to combine the inductances and capacitances of each SQUID into effective values and use Eq. 2.42 to describe the inductance change under an applied external magnetic field. Picture taken from [10].

Additional geometric inductances and capacitances are also included in the circuit diagram. These elements are generally not as spatially localised as the Josephson junctions; for an accurate description, they would need to be modelled as a series of small distributed elements. As the goal here is to obtain an analytical description of the system, these elements are combined into representative lumped elements wherever possible. This simplification comes at the cost of reduced accuracy.

From this point onward, we take the circuit shown in Fig. 3.3(b, c) as the basis and discuss further simplifications that are valid in different parameter regimes. Throughout the discussion, we consider a similar system: an upper (A) and a lower antenna pad (F), each connected to a SQUID loop via superconducting lines. The SQUID loops are connected to each other (C-D). The section between the junctions is referred to as the “island” in the following, as it is connected to the pads only through the junctions.

3.2.1 Simplification 1: No geometric inductances

For small loops and short connections, it is possible to describe the system in the simplest limit by neglecting all linear inductances. This treatment follows [10], to which the interested reader is referred for further details. The corresponding circuit is shown in Fig. 3.4.

In this limit, the Josephson junctions are the only relevant sources of inductance. Combining the inductances and capacitances of both junctions within a SQUID loop (from Fig. 3.4(a) to Fig. 3.4(b)) is only valid if no loop inductance is present. Without loop inductance, all junction inductances and capacitances are effectively connected in parallel, making them mathematically equivalent to a single lumped element. This allows the use of Eq. 2.42 to describe the SQUID inductance as a function of its parameters and the applied external magnetic flux.

The input impedance of this system is given by

$$Z_{\text{in}} = \frac{1}{iC_s\omega + \frac{1}{\frac{1}{iC_2\omega - \frac{i}{L_2\omega}} + \frac{1}{iC_1\omega - \frac{i}{L_1\omega}}}} - \frac{i}{C_c\omega}. \quad (3.18)$$

The resonance frequencies of the circuit are obtained by finding the zeros of the input impedance. In this case, the problem reduces to solving a quadratic equation in ω^2 , yielding

$$\omega_{\pm}^2 = \frac{\Omega_1^2 + \Omega_2^2}{2\beta} \pm \frac{\sqrt{\Omega_1^4 + \Omega_2^4 + 2\Omega_1^2\Omega_2^2(1 - 2\beta)}}{2\beta}. \quad (3.19)$$

Here, Ω_i is the bare eigenfrequency of SQUID loop i , including the capacitive contribution of the geometry,

$$\Omega_i^2 = \frac{1}{L_i(C_i + C_s + C_c)}. \quad (3.20)$$

It should be kept in mind that L_i is the effective inductance of SQUID loop i and depends on the external magnetic flux penetrating the loop, $L_i = L_i(\Phi_{i,\text{ext}})$, as defined in Eq. 2.42. The numerical factor β is defined as

$$\beta = 1 - \frac{(C_s + C_c)^2}{(C_1 + C_s + C_c)(C_2 + C_s + C_c)} \quad (3.21)$$

and describes the hybridisation of the bare SQUID modes. Inspection of Eq. 3.19 reveals two regimes. For strongly detuned SQUID eigenfrequencies and small β , the circuit frequencies remain close to the individual eigenfrequencies Ω_i . In contrast, when the eigenfrequencies are close and β is large, the modes hybridise strongly. In the special case $\Omega_1 = \Omega_2$, the modes split into an lower energy symmetric mode,

$$\omega_s = \frac{1}{\sqrt{(L_1 + L_2) \cdot \left(C_s + C_c + \frac{1}{\frac{1}{C_1} + \frac{1}{C_2}}\right)}}, \quad (3.22)$$

and an antisymmetric mode,

$$\omega_a = \frac{1}{\sqrt{\frac{1}{\frac{1}{L_1} + \frac{1}{L_2}} \cdot (C_1 + C_2)}}. \quad (3.23)$$

As can be seen from the expressions, the antisymmetric mode does not charge the coupling capacitance and therefore does not couple to external readout. An example of the resulting mode structure and spectrum for typical device parameters is shown in Fig. 3.5.

3.2.2 Simplification 2: With geometric inductances

For loop sizes as in the fabricated prototypes shown in Fig. 3.3(a) and 3.9(a, b), neglecting the linear inductance does not provide a good approximation of the system. This has two consequences. First, the linear inductances within the SQUID loops affect the phase drop across the junctions, which prevents the use of the analytical expression in Eq. 2.42.

The second effect is that geometric inductances also modify the mode structure and influence how capacitances can be treated. In particular, it is no longer possible to combine the inductances and capacitances of the system into effective lumped elements without further assumptions. The goal of this section is therefore to impose design restrictions that simplify the circuit model sufficiently to allow an efficient calculation of the resonance frequencies.

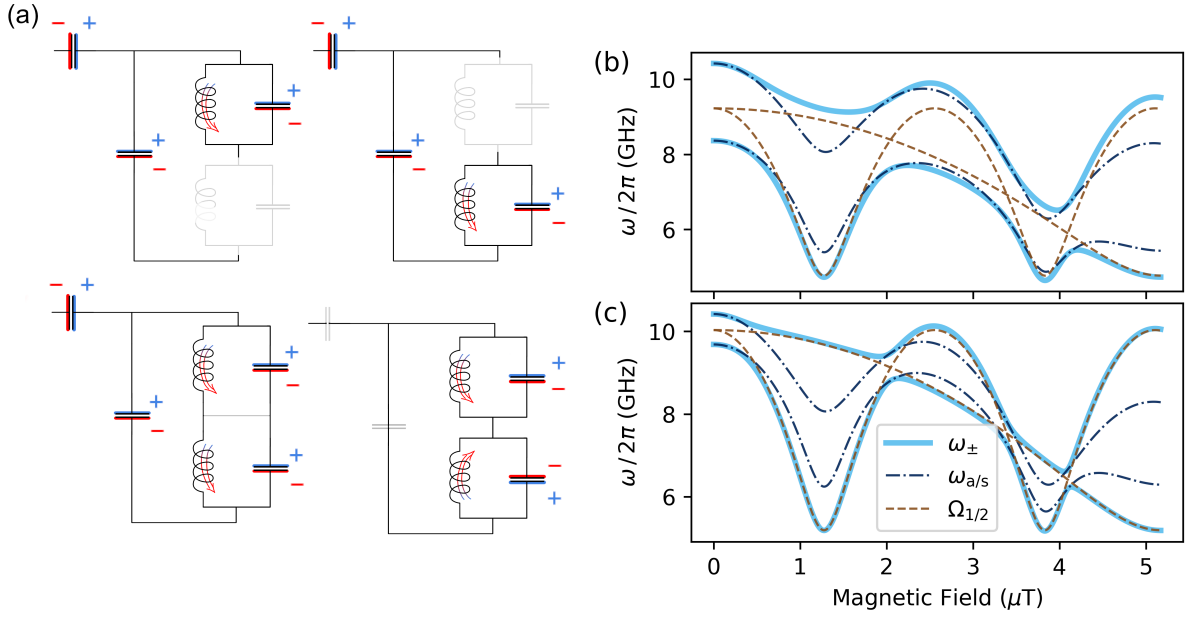


Fig. 3.5 Mode structure and spectrum of the simple model. (a) Mode structure of the individual SQUIDs, as well as the hybridised symmetric and antisymmetric modes. (b, c) Spectrum of the double SQUID obtained using Eqs. 3.19 and 2.42 for typical circuit parameters. The two measurable modes are shown as $\omega_{\pm}$. To understand their tuning, it is useful to consider the underlying bare SQUID frequencies Ω_1 and Ω_2 . In the strongly detuned regime, the circuit modes remain close to these bare frequencies. In contrast, when $\Omega_1 \approx \Omega_2$, the modes hybridise and the spectrum is more appropriately described in terms of the symmetric mode ω_s and the antisymmetric mode ω_a . Parameters used: $C_1 = C_2 = 500$ fF, $L_1 = L_2 = 450$ pH, $\alpha_I^1 = \alpha_I^2 = 0.27$, small-loop area $A_1 = 200 \mu\text{m}^2$, area ratio $k \approx 4$. In (b), the shunt capacitance is large, $C_s + C_c = 140$ fF, while in (c) it is smaller, $C_s + C_c = 40$ fF, leading to reduced hybridisation. The symmetric mode ω_s corresponds to the lower-frequency branch.

First assumption: Dominating capacitance

The circuit contains three types of capacitances. The first type consists of the shunt and coupling capacitances C_s and C_c , which enter the calculation as a sum. The second type is the junction capacitance C_J , which strongly depends on the junction size. In addition, there is a geometric capacitance C_{geo} between the pads and the island. The latter two capacitances are similar in nature but are not identical within the circuit model. This motivates the first design restriction:

$$C_{\text{geo}} \ll C_J \quad \text{or} \quad C_{\text{geo}} \gg C_J. \quad (3.24)$$

This assumption allows the smaller capacitance to be neglected, as illustrated in Fig. 3.6(a $\rightarrow$ b). In both limits, the circuit reduces to the same effective L - C topology. The difference lies only in the numerical values of the elements, e.g. if $L_{i,b}$ contributes to $L_{\text{var}, i}$ or $L_{\text{geo}, i}$. More details will be elaborated in Sec. 3.2.3.

To obtain the resonance frequencies of the circuit model under this simplification, the roots of the total circuit impedance must be determined. This leads to a fifth-order polynomial in ω^2 ,

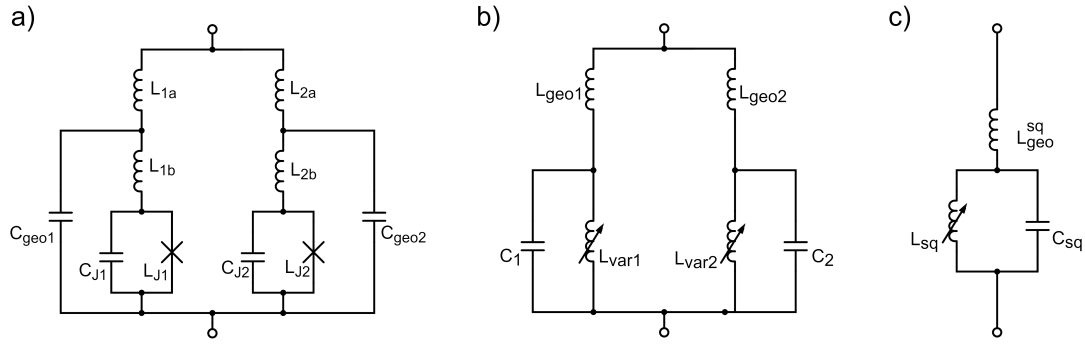


Fig. 3.6 SQUID-loop circuit simplification for a model including geometric inductance. (a) General circuit model of a single SQUID loop. Linear inductances are approximated by localised lumped elements. (b) Simplification for the case of a single dominating capacitance. Depending on whether C_J or C_{geo} dominates, the values of L_{var} and L_{geo} change, while the circuit topology remains the same. (c) Simplification for symmetric SQUID branches.

which has no closed analytical solution. While numerical solutions are in principle possible, they are undesirable here because the resonance frequency enters the fitting function used to extract system parameters from the measurements. The function must be evaluated for every data point and for each iteration of the optimisation routine, and therefore needs to be computationally efficient. Moreover, we are primarily interested in the low-frequency and coupling modes, not in all five possible solutions. This motivates the second restriction.

Second assumption: Symmetric SQUID loops

The second restriction requires the SQUID loops to be symmetric. Symmetry here means that the geometry of the left and right branches of each SQUID loop is identical. This requirement is not particularly restrictive, as the proposed designs are symmetric by construction. In addition to the geometry, the junctions themselves should also be identical. Ideally, both the capacitance and the inductance (or Josephson energy) are equal in the two arms of a SQUID.

In practice, the parameters will not be exactly identical, but the differences should be small, since the junctions are fabricated close to each other and undergo all fabrication steps in the same run. Because the SQUID loop is neither part of another closed loop nor connected to an external current source, the only DC currents that can flow are circulating currents within the loop. As a result, both junctions experience the same DC current, modifying their inductances in the same way.

With identical branches, the two sides of the SQUID loop have the same impedance. When connected in parallel, the effective impedance is therefore half of the branch impedance, following Eq. 2.10a. This simplification of the circuit model is illustrated in Fig. 3.6(b $\rightarrow$ c).

Resulting circuit

The resulting circuit after applying both simplifications is shown in Fig. 3.7. This circuit closely resembles the one without linear inductance shown in Fig. 3.4, but with several key differences.

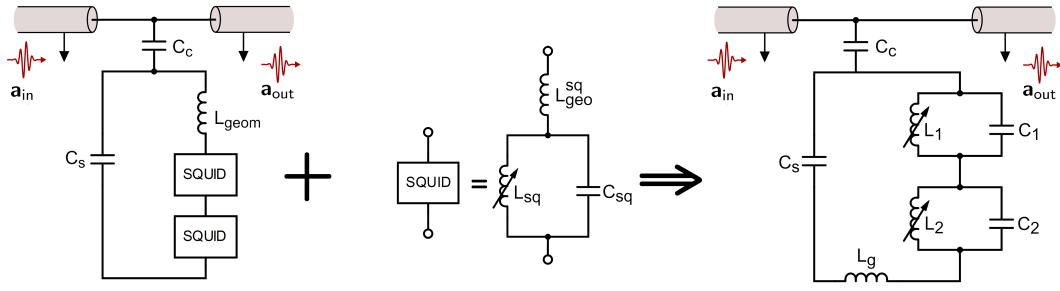


Fig. 3.7 Simplified circuit including geometric inductance. The resulting inductance is given by $L_g = L_{\text{geom}} + L_{\text{geo}}^{\text{sq},1} + L_{\text{geo}}^{\text{sq},2}$.

The most obvious addition is the geometric inductance L_g in series with the SQUID loops. This inductance accounts for the connection between the loops and may also include part of the loop inductance itself,

$$L_g = L_{\text{geom}} + L_{\text{geo}}^{\text{sq},1} + L_{\text{geo}}^{\text{sq},2}.$$

How much of the SQUID-loop inductance is absorbed into this lumped element depends on the arrangement of the capacitances.

The inclusion of L_g modifies the circuit impedance. In this case, determining the resonance frequencies reduces to solving a cubic equation in ω^2 , resulting in three modes. For realistic parameter values, however, the additional mode lies at much higher frequencies than the other two and typically falls outside the accessible microwave measurement range. Consequently, we focus only on the two lower-frequency modes. The analytical expressions for these modes are more involved than Eq. 3.19 and explicitly depend on L_g . The impedance is therefore solved analytically and used directly as a fitting function. For further details, the interested reader is referred to Sec. 7.1 in the appendix, where the functions are explicitly shown.

The second key difference concerns the SQUID inductance L_i . While in the simplified case it can be derived analytically, this is no longer possible when linear loop inductance is present, as discussed in Sec. 2.3.2. In addition to the junction inductance, the linear inductance of the SQUID loop must be included, with its contribution depending on the capacitance configuration. In the next section, we discuss how the resonance frequencies are used in a fitting function to extract the system parameters.

3.2.3 Fitting function and parameters

In this section, the considerations of the previous sections are combined into a single function that describes the change in resonance frequency of a double-SQUID system as the flux through the loops is swept. The resonance frequencies as functions of the circuit parameters are obtained analytically by finding the roots of the circuit impedance. The result is a set of two functions, each describing the frequency of one mode. In the case of negligible linear inductance, these functions are given by Eq. 3.19. The corresponding expressions for the case including linear inductance are provided in sec. 7.1 in the appendix.

The input parameters of the frequency function are the capacitances and inductances of the

circuit. These are fixed parameters, with the exception of the SQUID (or junction) inductances, which depend on the magnetic flux through the loop, $L_i(\Phi_{\text{ext}}^i)$. The flux itself is a function of the external magnetic field B_{ext}^i . In the experiment, the magnetic field at the sensor is not known directly; instead, the experimentally controlled parameter is the current applied to the coils that generate the magnetic field. We relate these quantities via

$$\Phi_{\text{ext}}^1 = \frac{I_{\text{ext}} - I_{\text{off}}}{I_p \cdot k}, \quad (3.25)$$

$$\Phi_{\text{ext}}^2 = \frac{I_{\text{ext}} - I_{\text{off}}}{I_p}. \quad (3.26)$$

Here, I_{ext} is the applied current through the coil, I_{off} is the current required to compensate the offset field, and k is the area ratio between the loops. As discussed in Sec. 3.1.2, what is effectively measured is the ratio between the magnetic fluxes through the loops, $k = \Phi_1/\Phi_2 \approx A_1/A_2$, which coincides with the geometric area ratio only in the case of a homogeneous tuning field.

The conversion factor I_p relates the coil current to a flux quantum threading the second SQUID loop. It is treated as a fitting parameter that can be extracted from the data and allows the conversion of the applied (or offset) current into an equivalent magnetic field using the area of the SQUID loop A_2 :

$$B_{\text{ext}} = I_{\text{ext}} \frac{\Phi_0}{I_p \cdot A_2} = \frac{I_{\text{ext}}}{I_p} \cdot T_2^{\text{B}} = \frac{I_{\text{ext}}}{I_p \cdot k} \cdot T_1^{\text{B}}. \quad (3.27)$$

Here, T_i^{B} denotes the magnetic-field period of SQUID loop i . To convert from current to magnetic field, the absolute area of the SQUID loop must be known; any uncertainty in this area directly translates into an uncertainty in the extracted offset field.

The final ingredient required to complete the fitting function is the dependence of the SQUID loop inductance on the external flux. Since L_i enters the analytical expression for the resonance frequencies and must be evaluated repeatedly during the fitting process, its computation must be efficient. In the case of negligible loop inductance, the analytical expression in Eq. 2.42 can be used directly, yielding $L_i = L_{\text{SQUID}}(\Phi_{\text{ext}}^i, I_0^{\text{SQUID}}, \alpha_I)$.

The situation is more involved when linear loop inductance is present. Rather than optimising a numerical solver for the implicit equation 2.46, we precompute the normalized SQUID inductance on a dense grid $L_{\text{norm}}(\beta_L, \Phi_{\text{ext}})$ once and replace the expensive implicit solving during the fit by a fast two-dimensional interpolation. The relevant parameter space is only two-dimensional, in contrast to the general case $L_{\text{SQUID}}(L_0^{\text{JJ}}, \beta_L, \Phi_{\text{ext}}, \alpha_I)$. This reduction is possible because we restrict the model to symmetric SQUID branches, which implies $\alpha_I = 0$. Also, the junction inductance L_0^{JJ} can be removed from L_{SQUID} , as it is just a multiplicative factor $L_{\text{SQUID}}(L_0^{\text{JJ}}, \beta_L, \Phi_{\text{ext}}) = L_0^{\text{JJ}} \cdot L_{\text{norm}}(\beta_L, \Phi_{\text{ext}})$

Specifically, a numerical simulation of a symmetric SQUID with linear loop inductance is performed, and the inverse (dimensionless) inductance $L_{\text{norm}}^{-1}(\beta_L, \Phi_{\text{ext}})$ is tabulated over the relevant parameter range. Storing the inverse quantity is advantageous because it remains well behaved even near points where the inductance becomes large. The resulting surface is shown in Fig. 3.8. From this lookup table, a bicubic spline surface is constructed using `scipy.interpolate.RectBivariateSpline`. This spline model can be evaluated in microseconds and is sufficiently smooth for use in least-squares fitting. The relative interpolation error

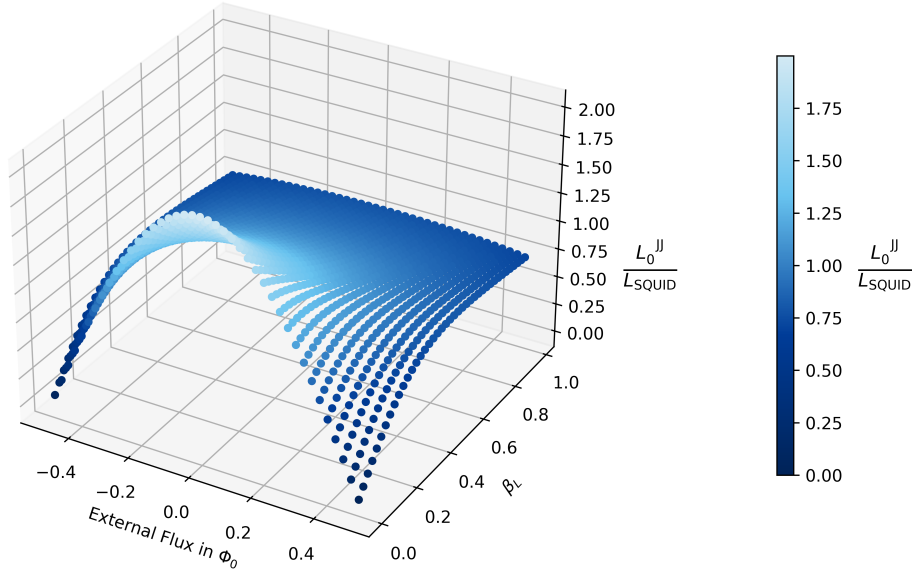


Fig. 3.8 3D plot of the SQUID inductance $L_{\text{norm}}(\beta_L, \Phi_{\text{ext}}, \alpha_I = 0)$. The point cloud is approximated using a spline model to enable fast evaluation of the inductance.

was estimated on an independently sampled grid and does not exceed $1.2 \cdot 10^{-5}$ and is significantly smaller in most regions of the parameter space.

Furthermore, we allowed the fit function to correct the squid inductance by a constant offset $L_i = L_{\text{SQUID}}(L_0^{\text{JJ}}, \beta_L, \Phi_{\text{ext}}) + L_{\text{corr}}$. The parameter L_{corr} should compensate for the fact that in our model we need the inductance parallel to the main source of capacitance, as indicated in Fig. 3.6. In the case of junction capacitance dominating, we need to subtract the loop inductance accounted for in $L_{\text{SQUID}}(L_0^{\text{JJ}}, \beta_L, \Phi_{\text{ext}})$, as this part is not in parallel to the junction capacitance. The other way around, if the main source of capacitance comes from the geometry, it may occur that the leads connecting to the capacitance carry additional inductance, thus we allow positive and negative values for L_{corr} . This may not be intuitive now, but will become clear when looking at the specific design of the second prototype in the next section.

3.3 Prototype design goals

Two different prototypes were fabricated based on the same double-SQUID principle but with different designs. The second design was intended to compensate for the weak points identified in the first one. Both designs are shown in Fig. 3.9. In further discussion we will label the first prototype P1 and the second prototype as P2. The prototypes are intended to be mounted in a two-dimensional waveguide, as shown in Fig. 5.3. The microwaves travel through the waveguide and interact with the sample while passing. More details can be found in Sec. 5.1.2.

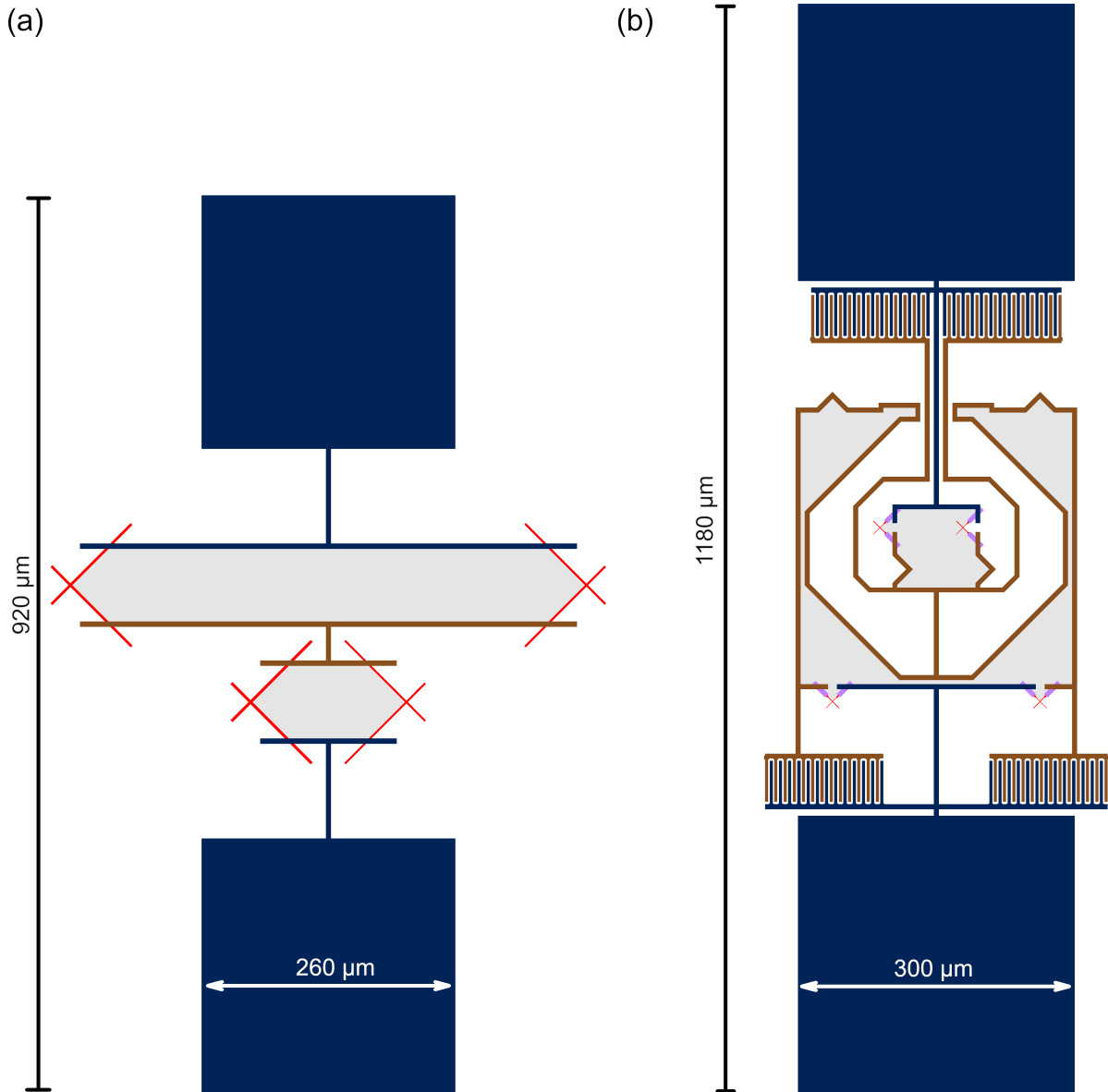


Fig. 3.9 Designs of the two fabricated prototypes: first design P1 in (a) and second design P2 in (b). The scale is identical for both devices. The antenna pads are shown in blue, the central "island" in brown, the junction leads in red and bandaging pads in purple. The SQUID loop areas are highlighted in light gray.

First design

The first design was kept intentionally simple to prove the principle and to avoid complications that could arise from more complex structures. The antenna pads have a square shape, and their size was estimated using HFSS simulations to achieve a reasonable coupling of 2 MHz of the lower mode to the waveguide. The loop areas were designed as $A_1 = 39210 \mu\text{m}^2$ and $A_2 = 9610 \mu\text{m}^2$, resulting in an area ratio of $k = 4.097$. The spacing between the SQUID loops was $40 \mu\text{m}$, and the centers of the two loop areas were separated by $120 \mu\text{m}$.

Following the approach of [10], the junctions were designed to provide the dominant capacitance of the circuit and therefore had to have a relatively large area. In addition, the junctions within each SQUID were intentionally designed with different sizes. The motivation for this asymmetry is that, for asymmetric junctions, the maximum SQUID inductance is limited, as illustrated in Fig. 2.8(a) for $\alpha_I \neq 0$. With a limited inductance, the resonance frequencies do not approach zero. The intention was to keep the lowest points of the spectrum within the measurable frequency range in order to extract the maximum amount of information from the measurement. The larger junction was designed with an area of $2.6 \times 2.6 \mu\text{m}^2$, limited by the fabrication process, while the smaller junction had an area of $1.8 \times 1.8 \mu\text{m}^2$.

Note:

When designing this prototype, the intention was to describe the SQUID loop inductance using Eq. 2.42. During the analysis of the measured data, however, it became evident that the linear (geometric) loop inductance is too large to be neglected. Since the junctions within each SQUID are intentionally asymmetric, none of the simplifications discussed in the previous sections are applicable to this device. As a pragmatic workaround, the SQUID inductance was therefore modified by adding a linear inductance in series,

$$L_{\text{SQUID}}^{\text{modi}}(\Phi_{\text{loop}}) = \frac{\hbar}{4eI_0^{\text{JJ}} \cos\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right) \sqrt{1 + \alpha_I^2 \tan^2\left(\pi \frac{\Phi_{\text{loop}}}{\Phi_0}\right)}} + L_{\text{lin}}. \quad (3.28)$$

This description is not physically rigorous, as it neglects the self-consistent interaction between circulating currents and the loop inductance. Nevertheless, as long as the added linear inductance remains smaller than the effective Josephson inductance, the resulting inductance modulation is qualitatively similar to that of a full model including loop inductance. For the purpose of interpreting the measurements of the first prototype, this approximation was found to be sufficient. Since the first prototype served only as a temporary proof-of-principle device and was not intended for further optimisation or reuse, the effort required to develop a fully self-consistent description of the asymmetric SQUID with finite loop inductance was deliberately omitted.

Second design

The second design was planned and fabricated after the first prototype had been thoroughly measured and characterized. It was intended to improve upon the following weaknesses:

- **Stronger coupling**

The first prototype was undercoupled, mainly due to a poor internal quality factor. To increase the coupling, both the size of the antenna pads and their spacing were increased.

The goal was to enhance the coupling strength by approximately a factor of two to 4 MHz, as estimated from HFSS simulations. The stronger coupling was expected to improve, in particular, the visibility of the higher-frequency mode.

- **Capacitance relocated away from the junction**

The first prototype exhibited coupling to two-level systems over the entire frequency range. As discussed in more detail in Chapter 5, these two-level systems were suspected to reside in the oxide layer of the junctions [24, 25]. To mitigate this effect, an additional source of geometric capacitance was introduced into the circuit. Each of the four comb-shaped elements shown in Fig. 3.9(b) provides a capacitance of approximately 100 fF, as estimated from HFSS simulations. At the same time, the junction capacitance was reduced by decreasing the junction area to approximately $0.4 \times 0.4 \mu\text{m}^2$, reducing the total junction area of a SQUID by a factor of about 60.

- **Symmetric SQUIDs**

To satisfy the requirements for the circuit simplification discussed in Sec. 3.2.2, the junctions within each SQUID were designed to have identical areas.

- **Junction orientation**

The junctions were fabricated using the Manhattan process, as discussed later in Chapter 4. Imaging of small test junctions revealed that reproducible fabrication is only achieved when the junctions are oriented along two orthogonal directions. Consequently, the junctions in the second design are oriented to the left and downward, as shown in Fig. 3.9(b).

- **Gradient insensitivity**

As shown in Sec. 3.1.2, magnetic field gradients can introduce significant uncertainties into the measurement. The second design was therefore intended to mitigate gradient effects. To illustrate this, consider the magnetic flux through a SQUID loop. In general, the normal component of the magnetic field $\vec{B}_{\text{ext}}$ must be integrated over the effective loop area A :

$$\Phi_{\text{ext}} = \int_A \vec{B}_{\text{ext}}(\vec{x}) \cdot \vec{n} dA = \int_A B_{\text{ext}}^n(\vec{x}) dA. \quad (3.29)$$

Here, $\vec{n}$ is the unit normal vector of the loop, and $B_{\text{ext}}^n(\vec{x})$ denotes the normal component of the magnetic field. In general, $B_{\text{ext}}^n(\vec{x})$ can be an arbitrary function of position, meaning that two loops experience the same flux only if their areas and positions coincide exactly.

This changes if the magnetic field varies only linearly in space, which is a good approximation when the loop size is small compared to the distance to the magnetic field source. Without loss of generality, we choose a coordinate system in which the loop lies in the xy -plane with $\vec{n} = \vec{e}_z$. The magnetic field can then be written as

$$B_{\text{ext}}^z(x, y) = B_0 + B_x x + B_y y. \quad (3.30)$$

Using this expression, the magnetic flux becomes

$$\Phi_{\text{ext}} = \int_A B_{\text{ext}}^z dA = B_0 \cdot A + B_x \int_A x dA + B_y \int_A y dA. \quad (3.31)$$

The integrals correspond to the definition of the centroid (center of area) of a two-dimensional region:

$$\bar{x} = \frac{1}{A} \int_A x dA, \quad \bar{y} = \frac{1}{A} \int_A y dA. \quad (3.32)$$

This yields

$$\Phi_{\text{ext}} = A [B_0 + B_x \bar{x} + B_y \bar{y}] = A \cdot B_{\text{eff}}(\bar{x}, \bar{y}). \quad (3.33)$$

In words, for a linearly varying magnetic field, the effective magnetic field experienced by a SQUID loop depends only on the coordinates of its center of area. If two SQUID loops share the same center of area, they experience the same effective magnetic field. In the second prototype, the outer loop surrounds the inner one, and both loops share the same center position, rendering the design first-order insensitive to magnetic field gradients. Higher-order gradients and fabrication-induced misalignments can still introduce residual effects.

The spacing between the SQUID loops was intentionally chosen to be large in order to minimize interactions between (circulating) currents in the loops. The junctions are fabricated in a separate step with limited alignment accuracy compared to the main structure, which is patterned in a single lithography step. For small misalignments of the junctions, the area of the inner loop remains invariant, as any area increase on one side is compensated by a corresponding decrease on the opposite side. The loop areas of the second design were planned to be $A_1 = 32840 \mu\text{m}^2$ and $A_2 = 8100 \mu\text{m}^2$, resulting in an area ratio of $k = 4.055$.

CHAPTER 4

Fabrication

All measured devices were fabricated in the QNZT cleanroom. The aim of this chapter is to explain the fabrication process as well as the relevant design considerations. It is also meant as a guideline for anyone who intends to fabricate a similar device.

4.1 Fabrication techniques

In order to fabricate the magnetic field sensors presented in this work, it is necessary to create micrometer- to sub-micrometer-scale structures such as Josephson junctions and superconducting geometries for antenna pads or interconnections. These structures are patterned on a carrier substrate, in our case silicon, using two-dimensional lithography processes carried out in several steps. For a size comparison, a picture of a sample chip is shown in Fig. 4.1.

Lithography is used to create a stencil on the substrate. The general working principle is as follows:

1. Resist coating

At the beginning, the substrate is coated with an organic polymer layer of homogeneous thickness, referred to as resist. This is usually done by depositing the liquid resist onto the substrate and subsequently rotating it at high speed (spin coating). This process results in a nearly homogeneous resist layer, whose thickness depends on the chosen resist type and rotation speed. After spinning, the solvent must be removed to solidify the resist, which is typically achieved by heating (soft bake).

2. Exposing

Each resist is designed to be sensitive to a specific type of radiation. Exposing the resist to this radiation locally changes its chemical structure. The modified regions exhibit different

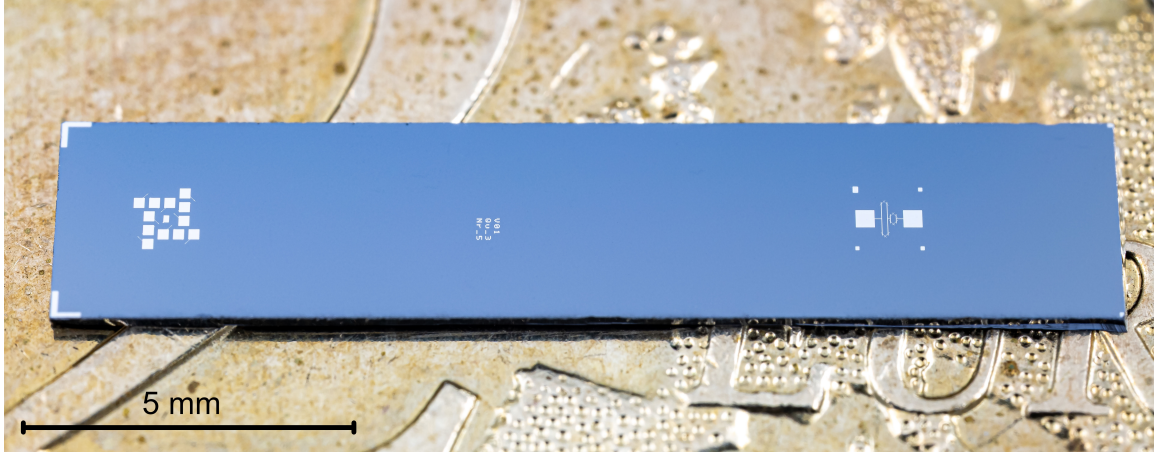


Fig. 4.1 Picture of a silicon chip with the first prototype. The chip has dimensions of $15 \text{ mm} \times 3 \text{ mm}$ and is positioned on a two-euro coin for size comparison. On the left side, there is a structure to estimate the junction resistance, while on the right side the sensor is placed. Picture taken by D. Jordan.

properties, which are exploited in the subsequent development step. In this work, both photoresist and electron-beam resist were used. Photoresist changes its structure when exposed to blue or ultraviolet light, whereas electron-beam resist requires high-energy electrons to modify the molecular structure.

While the underlying exposure mechanism is similar for both methods, there is a key difference in achievable structure size and exposure speed. For prototype fabrication, exposure is typically performed using a single radiation beam that is deflected and switched on and off by the system to expose the desired pattern. The minimum achievable beam spot size is limited by the system optics and, ultimately, by the diffraction limit.

For an optical Gaussian beam (TEM_{00} mode), the beam waist is given by

$$w_0 = \frac{2\lambda}{\pi\Theta}, \quad (4.1)$$

where λ is the wavelength and Θ is the full angular divergence of the beam in radians [26]. Technically, Θ is limited to values smaller than π , resulting in a minimum spot size on the order of the wavelength, typically around hundreds of nanometers for optical lithography.

For electron-beam lithography, a similar Gaussian approximation can be made using the de Broglie wavelength $\lambda = h/p$, with Planck's constant h and the electron momentum p . In this case, the resolution limit is on the order of 10 nm for a 30 kV beam. The exposure time for a fixed-size pattern is limited by the radiation intensity, which is typically inversely related to the achievable resolution. This makes optical lithography preferable for large structures, while electron-beam lithography is necessary for small and precise features.

3. Developing

In the development step, the difference in chemical structure induced by exposure is used to selectively remove parts of the resist. Depending on the resist type, either the exposed or the unexposed regions are dissolved in the developer. In positive resists, the exposed material is removed, whereas in negative resists the unexposed regions are dissolved. After development, a detailed stencil of the desired structure remains on the substrate.

The resist mask created during lithography can be used in different ways. In this work, two different pattern transfer methods were employed:

- **Lift-off process**

In the lift-off process, the resist mask is applied directly to the bare substrate. After exposure and development, a metal layer is deposited over the entire surface. The metal adheres to the substrate in the openings of the resist stencil and covers the resist elsewhere. In the subsequent lift-off step, the remaining resist is dissolved, lifting off the metal deposited on top of it.

Depending on the deposition angle and the exact resist profile, unwanted connections between metal on the substrate and metal on the resist can occur. These connections may tear off intended structures or leave behind unwanted metallic residues. To mitigate this issue, an undercut profile can be created by using a bilayer resist system in which the lower layer is more sensitive to radiation. This produces an overhanging resist profile, as illustrated in Fig. 4.2(c). The lift-off process enables techniques such as the Manhattan process for junction fabrication, discussed in more detail in Sec. 4.3.

- **Subtractive process**

In the subtractive process, the metal layer is deposited uniformly on the wafer prior to resist application. After lithography and development, the resist mask remains on top of the metal. The next step is etching, where the exposed metal is removed using chemically aggressive solutions (wet etching) or ionized gases (dry etching). The etchants are chosen such that the metal is removed while the resist remains intact, protecting the underlying regions.

In a subsequent step, the resist itself is removed using a different etching or stripping process that leaves the metal unaffected. This process is schematically illustrated in Fig. 4.2. Compared to lift-off, the subtractive process is generally more robust and compatible with a wider range of materials. However, it can damage delicate substrates, and metals that are difficult to etch chemically, such as gold or platinum, are often unsuitable.

The main reason for employing this method in the present work was its robustness for closely spaced structures, such as the capacitive elements shown in Fig. 3.9(b). Attempts to fabricate these structures using the lift-off process were unsuccessful because the undercut in the resist became too large. As a result, parts of the top resist layer were effectively suspended and mechanically unstable, preventing further reliable processing. In Fig. 4.3 images of the different masks for these capacitive elements are shown for comparison.

4.2 Machines

The following equipment was used for the fabrication:

- **E-beam lithography**

Electron-beam lithography was performed using a *Raith eLINE Plus* system with a working area of 100 mm × 100 mm. The system is equipped with a laser interferometer for high-precision stage positioning. The maximum electron energy is 30 keV, and the available gun

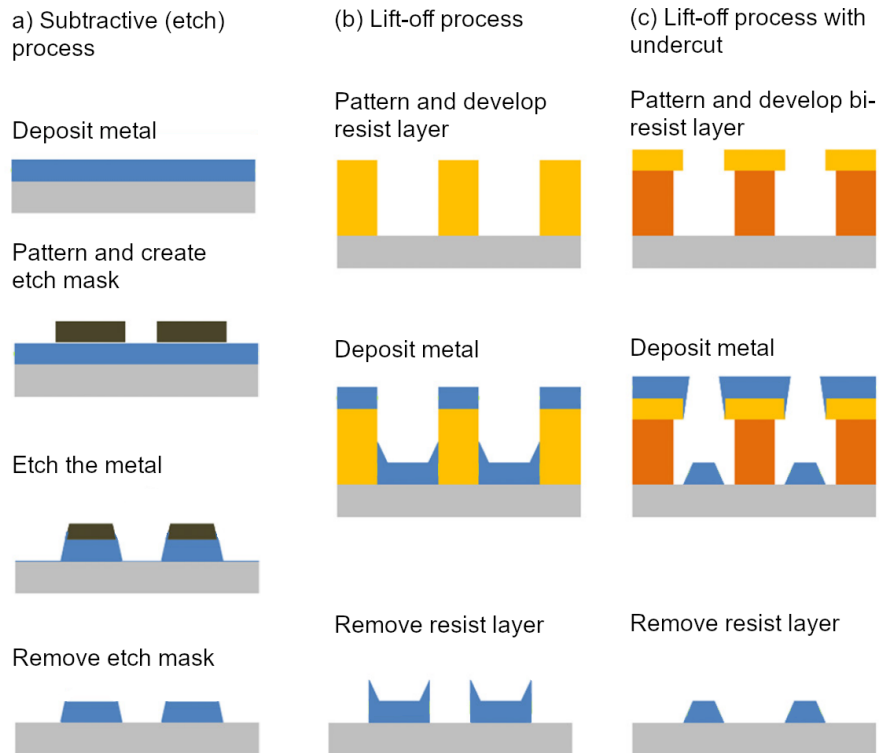


Fig. 4.2 Schematics of the nanofabrication workflow used in this project: (a) subtractive (etch) process, (b) lift-off process, and (c) lift-off with undercut. Figure taken from [27] and modified.

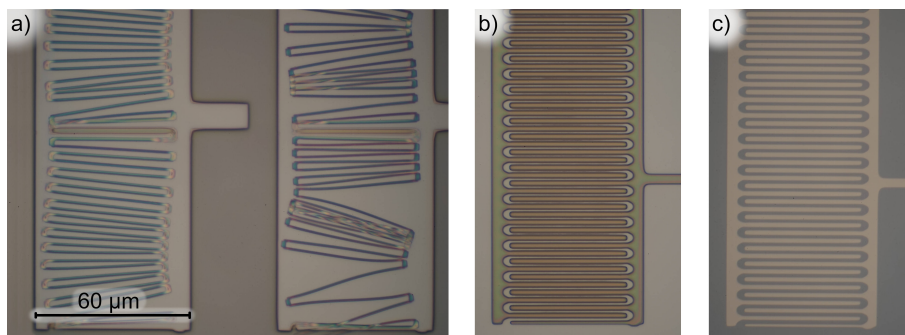


Fig. 4.3 Optical images of the capacitive elements. (a) Resist mask for the lift-off process on a silicon substrate. Due to the large undercut, the thin leads of the mask became mechanically unstable. (b) Resist mask for the subtractive process on niobium. The feature sizes are just large enough to be fabricated using optical lithography. (c) After etching the mask shown in (b), a niobium structure remains on the silicon substrate.

apertures are 7, 10, 30, 60, and 120 μm . For junction fabrication, the 10 μm aperture was used, resulting in a typical beam current of approximately 38 pA.

- **Optical lithography**

Optical lithography was carried out using a *Microtech Laserwriter LW405D*, a direct-write, maskless lithography system. The light source operates at a wavelength of $\lambda = 405 \text{ nm}$ and the system is equipped with four different lenses as well as an automated focus control. We used the lens with the second-highest resolution, labeled as *lens 4* in the system. This lens has a numerical aperture of 0.4 and allows the fabrication of structures with feature sizes down to approximately 2 μm .

The spacing between subsequent scan lines can be adjusted using the so-called *D-step* parameter. We chose the smallest available value for the selected lens, corresponding to a step size of 0.2 μm . These settings significantly reduced the writing speed but were necessary to achieve the resolution required for the fine capacitive elements in Fig. 4.3.

- **Deposition / Oxidation**

For Josephson junction fabrication, the *Plassys MEB550S* system (hereafter referred to as *Plassys*) was used. This is an electron-beam evaporation system equipped with multiple crucibles; in this work, it was used exclusively for aluminum deposition. The system includes an integrated ion gun for in-situ sample cleaning (etching/milling) using argon (Ar) plasma prior to deposition and supports in-situ oxidation.

The substrate stage can be rotated and tilted in a fully automated manner, enabling Manhattan-process junction fabrication explained in Sec.4.3. Oxidation was performed in a pure O_2 environment. This system was also used for depositing the bandaging layer to electrically connect the junctions to the ground structure in the second prototype. In this case, an etching step was applied prior to deposition to remove the native oxide layer from the metal surface.

- **Sputtering**

The *AJA ATC1800-HY* system (hereafter referred to as *AJA*) was used for additional metal deposition. This system is equipped with both electron-beam evaporation and magnetron sputtering capabilities. Magnetron sputtering was used to deposit niobium (Nb) for the subtractive lithography process of the second prototype. A 50 nm thick Nb layer was deposited, followed by a thin 5 nm tantalum (Ta) capping layer to prevent oxidation of the niobium surface.

The AJA system was also used to fabricate test junctions using the Manhattan process, providing an alternative to the Plassys system.

- **Etching**

Dry etching for subtractive processes was performed using the *Sentech SI 500*, an inductively coupled plasma (ICP) etching system. The system is equipped with a laser-based thickness monitoring setup to track layer thicknesses during etching.

CF_4 -based plasma etching was used to remove the Nb/Ta layers, with typical gas flows of 20 sccm CF_4 and 2 sccm O_2 and a chamber pressure of 1 Pa. After metal etching, the remaining resist was removed using an O_2 plasma with a gas flow of 60 sccm and a chamber pressure of 10 Pa.

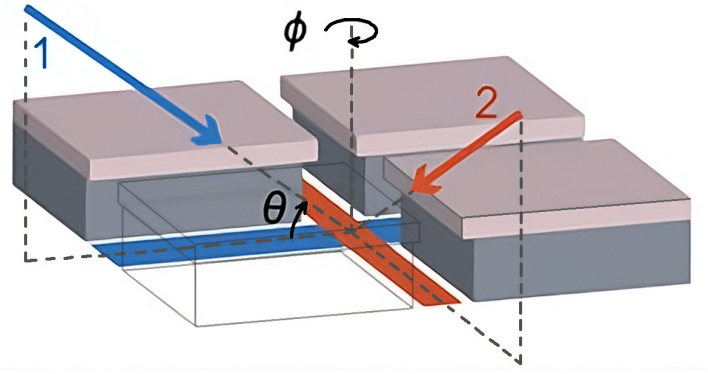


Fig. 4.4 Manhattan process for junction fabrication. The metal is evaporated at a shallow angle $\Theta \approx 30^\circ$. The first metal layer (blue) reaches only the substrate through the aligned slit in the resist. After oxidation, the azimuthal angle ϕ is rotated by 90° before the second metal evaporation. Graphic taken from [29] and modified.

4.3 Junction fabrication

A Josephson junction consists of two superconductors separated by a non-superconducting gap, as described in Chapter 2.2.2. While, in principle, any superconductor–insulator–superconductor combination can be used, a very common choice is aluminum as the superconductor and aluminum oxide as the insulating barrier, which was also used in this project. By carefully controlling oxidation time and pressure, the thickness of the oxide layer — and thus the junction characteristics — can be tuned reliably.

The junctions were fabricated using the Manhattan process [28], which is shown schematically in Fig. 4.4. In this process, the junction is formed using only a single resist mask and two metal deposition steps separated by an oxidation step. The use of a single mask is possible because the metal is deposited at an angle. For this to work, the resist thickness must be comparable to or larger than the junction width, and the junction leads must be oriented approximately perpendicular to each other.

Metal evaporated at a shallow tilt angle $\Theta \approx 30^\circ$ will reach the substrate only if the slit in the resist is oriented along the evaporation direction. For slits oriented perpendicular to the evaporation direction, the metal is deposited on the sidewalls of the resist but does not reach the substrate due to the shadow cast by the thick resist. After the first evaporation, the aluminum is exposed to an oxygen environment, forming the insulating oxide layer required for the Josephson junction. Subsequently, the second aluminum layer is deposited after rotating the sample by 90° .

4.3.1 Junction Parameters

The junction introduces a nonlinear inductance into the circuit. An estimate of the Josephson energy E_J can be obtained from room-temperature measurements of the normal-state resistance R_N of the junction. The relation between these two quantities is given by the Ambegaokar–Baratoff relation [30]

$$E_J = \frac{1}{2} \frac{\hbar \Delta}{(2e)^2} R_N^{-1}, \quad (4.2)$$

where h is Planck's constant, e the elementary charge, and Δ the superconducting energy gap of the material. The normal-state resistance of a thin insulating barrier between two conductors is governed by electrons tunneling through the barrier. The theory of tunneling shows that the tunnel probability — and thus the normal-state resistance — depends strongly on the barrier thickness and decreases exponentially with increasing thickness. The resistance of a junction R_J can be written as

$$R_J = \frac{R_{uA}}{A_J} \quad \text{with} \quad R_{uA} \propto e^{-d \cdot a}, \quad (4.3)$$

where R_{uA} is the unit-area resistance, A_J the junction area, d the oxide thickness, and $a > 0$ a parameter determined by the barrier potential.

From Eqs. 2.22 and 4.2, it follows that the junction inductance is proportional to the normal-state resistance. This allows the definition of a unit-area inductance L_{uA} such that $L_{j,0} = L_{uA}/A_J$. For a fixed material choice, L_{uA} depends only on the oxide thickness d .

The capacitance between the two electrodes of the junction can be approximated using a parallel-plate capacitor model, where the aluminum oxide acts as the dielectric spacer. The junction capacitance C_J can thus be estimated as

$$C_J = \frac{\epsilon_0 \epsilon A_J}{d} = C_{uA}(d) \cdot A_J, \quad (4.4)$$

where ϵ_0 is the vacuum permittivity and ϵ the relative permittivity of the oxide. As we saw before, for a fixed material choice, the unit-area capacitance C_{uA} depends only on the oxide thickness d again.

Aluminum exposed to an oxygen environment rapidly forms a native oxide layer, which subsequently inhibits further oxygen diffusion into the metal. This behavior is described by the Cabrera–Mott mechanism [31]. According to Ref. [32], for an oxygen pressure of 10 mbar at approximately room temperature, the oxide layer reaches a thickness of about 0.45 nm within the first 60 s. The growth rate then decreases significantly; during the following 15 min the thickness increases by only about 0.06 nm. Although the exact values depend on pressure, temperature, and the deposition of the second aluminum layer, this illustrates that the capacitance varies only weakly with oxidation time, whereas the resistance changes exponentially.

Parameter regime of the first prototype

In the first prototype, the goal was to make the junctions the dominant source of both inductance and capacitance. In the simplified model discussed in Section 3.2.1, the inductances and capacitances of the two junctions within a SQUID loop can be combined into effective values. Assuming that both junctions are fabricated in the same process step and therefore have (nearly)

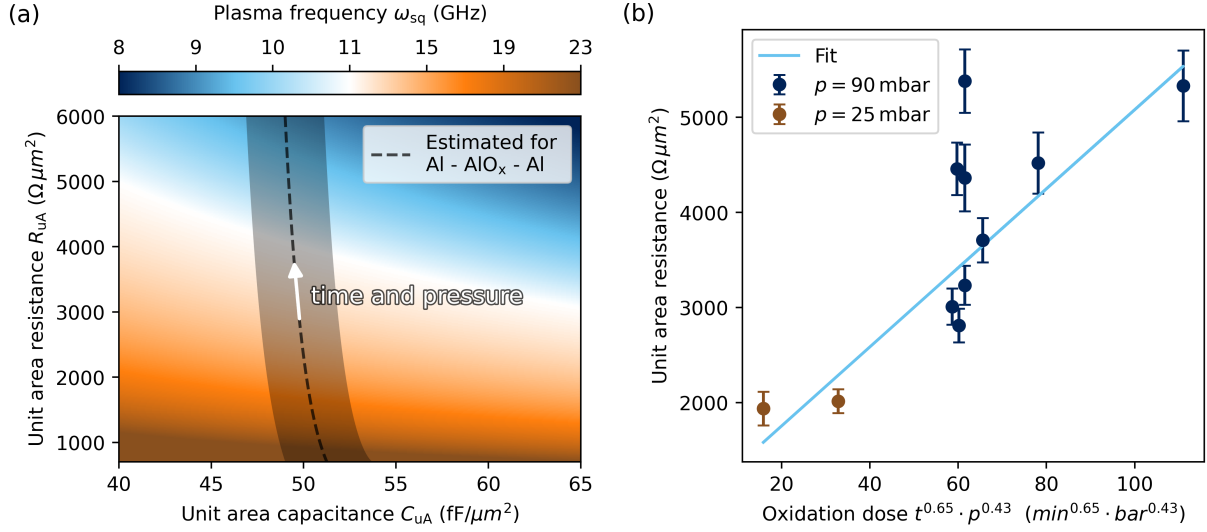


Fig. 4.5 (a) Plasma frequency as a function of normal-state resistance and unit-area capacitance, assuming a superconducting gap of $200 \mu eV$. Since both axes depend on the oxide thickness, only certain combinations are accessible by varying the oxidation dose, indicated by the gray line. (b) Measured unit-area resistance of 11 batches fabricated in separate oxidation runs. Each batch contained 8 to 16 samples. Motivated by Ref. [33], the x-axis shows the oxidation dose $t^{0.65}p^{0.45}$. Error bars include statistical variations within a fabrication batch and uncertainty in the junction area.

identical unit-area parameters (L_{uA}, C_{uA}), one finds

$$\omega_{sq}^{-2} = C_{sq} \cdot L_{sq} \quad (4.5)$$

$$= (C_{J1} + C_{J2}) \cdot \frac{1}{1/L_{J1} + 1/L_{J2}} \quad (4.6)$$

$$= C_{uA}(A_{J1} + A_{J2}) \cdot \frac{1}{A_{J1}/L_{uA} + A_{J2}/L_{uA}} \quad (4.7)$$

$$= C_{uA} \cdot L_{uA} \quad (4.8)$$

$$= C_J \cdot L_{j,0} = \omega_J^{-2}. \quad (4.9)$$

The plasma frequency ω_{sq} of the SQUID is therefore equal to the plasma frequency ω_J of an individual junction and depends only on the unit-area values C_{uA} and L_{uA} . Operating the device in a regime where the junction dominates both inductance and capacitance places the resonance frequency close to the junction plasma frequency, with additional circuit elements shifting it slightly to lower values.

The theoretical plasma frequency as a function of unit-area resistance and unit-area capacitance is shown in Fig. 4.5(a). As discussed above, resistance and capacitance are not independent but linked via the oxide thickness of the Al-AlO_x-Al junction. Since the capacitance was not measured directly, the exact position of the gray curve in Fig. 4.5(a) is uncertain and should be interpreted only as a qualitative guide. Overall, designing a SQUID where the junction dominates inductance and capacitance reduces to two key design choices:

- **Junction size**

The capacitance scales with junction area, so the junctions should be as large as possible for junction capacitance to dominate.

- **Oxidation dose**

The maximum achievable frequency is set by the junction plasma frequency, which depends primarily on the oxide thickness. The oxidation parameters must therefore be chosen to yield plasma frequencies of approximately 10 GHz.

We targeted a unit-area resistance of approximately $4000 \Omega \cdot \mu\text{m}^2$, corresponding to a plasma frequency of about 11.0 GHz, assuming a unit-area capacitance of $50 \text{ fF}/\mu\text{m}^2$ and a superconducting gap of $200 \mu\text{eV}$. This resistance is high compared to values typical of transmon junction fabrication [34]. Oxidation in this regime is relatively irreproducible. This is shown in Fig. 4.5(b), where the resistance results of several test runs are plotted. The error bars indicate the variation within one fabrication run. The result from different fabrication runs spread even more and we could not correlate this deviation to an external influence.

The junctions of the final first-prototype devices were fabricated using static oxidation in pure O_2 at a pressure of 90 mbar for 28 min. The O_2 inlet time was 4 min 41 s and the pumping time to reduce the pressure sufficiently to stop oxidation was 2 min 21 s, both in addition to the oxidation time. The large junctions of size $2.8 \times 2.8 \mu\text{m}^2$ exhibited normal-state resistances between 530Ω and 740Ω .

Parameter regime of the second prototype

In the second design, the capacitance was intentionally moved away from the junctions as discussed in Sec. 3.3. This allows the junction size and oxide thickness to be chosen to obtain a suitable junction inductance without significantly affecting the total capacitance. The oxidation parameters could therefore be selected closer to those used in standard junction fabrication, resulting in more reproducible resistance values.

For the final second-prototype devices, oxidation was performed at a pressure of 0.6 mbar with an oxidation time of 340 s. The O_2 inlet time was 66 s and the pumping time to stop oxidation was 67 s, both in addition to the oxidation time. Junctions with an area of $0.38 \times 0.38 = 0.1444 \mu\text{m}^2$ showed normal-state resistances of approximately 1070Ω .

Due to technical issues in the cleanroom, the AJA system was also tested for Manhattan-process junction fabrication. Owing to the sample mounting system, the azimuthal rotation angle ϕ lacked absolute calibration and had to be set manually, leading to an estimated error of about 2° . The 90° rotation between evaporations was machine-controlled with high precision, but the initial alignment error propagated to the second deposition. In the AJA system, the oxidation pressure was limited to 0.133 mbar ($= 100 \text{ mTorr}$). A dynamic oxidation process was used, maintaining a constant O_2 flow of 2 sccm after reaching the target pressure. With an oxidation time of 7 min (including 77 s inlet time and excluding a 4 s pump-down), junction resistances comparable to those obtained with the Plassys system were achieved. Ultimately, all final prototypes were fabricated using the Plassys system.

Tab. 4.1 Details of the fabrication process/parameters (junctions). The small junctions refer to the second prototype, while the big junctions refer to the first prototype.

	Description	Small Junctions	Big Junctions
Resist Layer 1	type spinning approximate thickness baking	MMA EL 13 3000 rpm, 100 s 800 nm 90 s @ 150 °C	
Resist Layer 2	type spinning approximate thickness baking	PMMA 950k A4 1000 rpm (100 s) 350 nm 3 min @ 150 °C	AR-P 672.11 2000 rpm (100 s) 2600 nm
Lithography	machine aperture dose development	Raith eLINE Plus 10 μ m, 30 kV 370 μ C/cm ² 400 μ C/cm ² 3 H ₂ O : 1 IPA, 2 min @ 6 °C	
Deposition	method material thickness max junction sidelength	evaporation (Plassys) Al + oxidation + Al 30 nm + 70 nm 700 nm	30 nm + 50 nm 2800 nm
Lift-off	fast slow (preferable)	Acetone @ 50 °C minimum 2 h Acetone @ 20 °C minimum 12 h	

4.4 Settings and Recipes

All devices were fabricated on two-side-polished 2 inch silicon wafers. The first prototype was fabricated entirely from aluminum using a lift-off process. In a first step, the junctions were fabricated using the Manhattan process, followed by the fabrication of the antennas and connections in a second step.

Due to the fine capacitive structures, the second prototype used a niobium ground structure with a thin tantalum capping layer, fabricated using a subtractive process. The junctions were added in a second step, followed by an additional bandaging layer to electrically connect them to the loops and antenna pads without introducing additional oxide interfaces.

Details of the lithography recipes and a collection of the most important fabrication parameters are summarized in Table 4.1 for the junctions and Table 4.2 for the other elements and stitching layers.

Tab. 4.2 Details of the fabrication process for optical lithography. The + sign in the development step indicates a change of beaker.

	Description	Ground Structure Subtractive process	Stitching Lift-off process
Deposition	method material thickness	DC magnetron sputter (AJA) Nb + Ta capping 50 nm + 5 nm	e-beam evaporation (Plassys) Al 120 nm
Resist	type spinning approximate thickness baking	AR-N 4340 4000 rpm (100 s) 1500 nm 1 min @ 100 °C	AR-P 5350 4000 rpm (100 s) 900 nm 4 min @ 105 °C
Lithography	machine optical settings dose baking after exposure development development time baking after development	Microtech LW405D Lens 4, D-step: 2 350 mJ/cm ² 3 min @ 120 °C AR-300-475 45 s + 30 s 3 min @ 120 °C	55 mJ/cm ² - 2 H ₂ O : 1 AR 300 35 60 s -
Nb-Etching	gases power chamber pressure time (remove + over etch)	20 sccm CF ₄ + 2 sccm O ₂ RF power 50 W, ICP power 50 W 1 Pa 55 s + 30 s	
Resist-Etching	gases power chamber pressure time (remove + over etch)	60 sccm O ₂ RF power 50 W, ICP power 50 W 10 Pa 120 s + 600 s	
Lift-off	fast slow (preferable)	Acetone @ 50 °C minimum 2 h Acetone @ 20 °C minimum 12 h	

CHAPTER 5

Measurements and Characterisation

In this chapter we present all measured properties of both prototypes. The chapter begins with a description of the experimental setup. We then show the device responses and extract their characteristic parameters. This includes a comparison between the designed parameters and the values obtained from measurements.

Afterwards, we address the primary purpose of the device, namely magnetic-field sensing. This includes offset-field measurements, magnetic shield attenuation, and noise characterisation. Whenever the same measurement was performed for both prototypes, the results are shown and compared directly.

Instead of collecting all discussion and conclusions at the very end of the chapter, the interpretation of the results is included at the end of each section to improve readability.

5.1 Experimental setup

In this section we describe the experimental platform used to characterise the sensors. The devices are fabricated from aluminium and niobium. To reach superconductivity, the temperature must be below the critical temperature of aluminium, $T_c = 1.19\text{ K}$ (see Table 2.1).

All measurements were performed at significantly lower temperatures, typically in the range of 20 to 40 mK. Operating well below the critical temperature ensures stable conditions, reduces the influence of local heat sources, and suppresses thermal population of the electromagnetic modes.

In addition to cryogenic cooling, a carefully designed microwave setup is required to guarantee low-noise input signals and low-noise amplification of the transmitted signal.

5.1.1 Cryostat and wiring

Two different dilution cryostats were used for the experiments. The first was an Oxford Triton 400, a bottom-loading design with the sample mounted at the bottom, as shown in Fig. 5.1(a) [35]. The second was a Qiniu SL-101-Pro, a more compact upright version with the sample mounted near the top and optimised for quick cooling and warming cycles, shown in Fig. 5.1(b) [36].

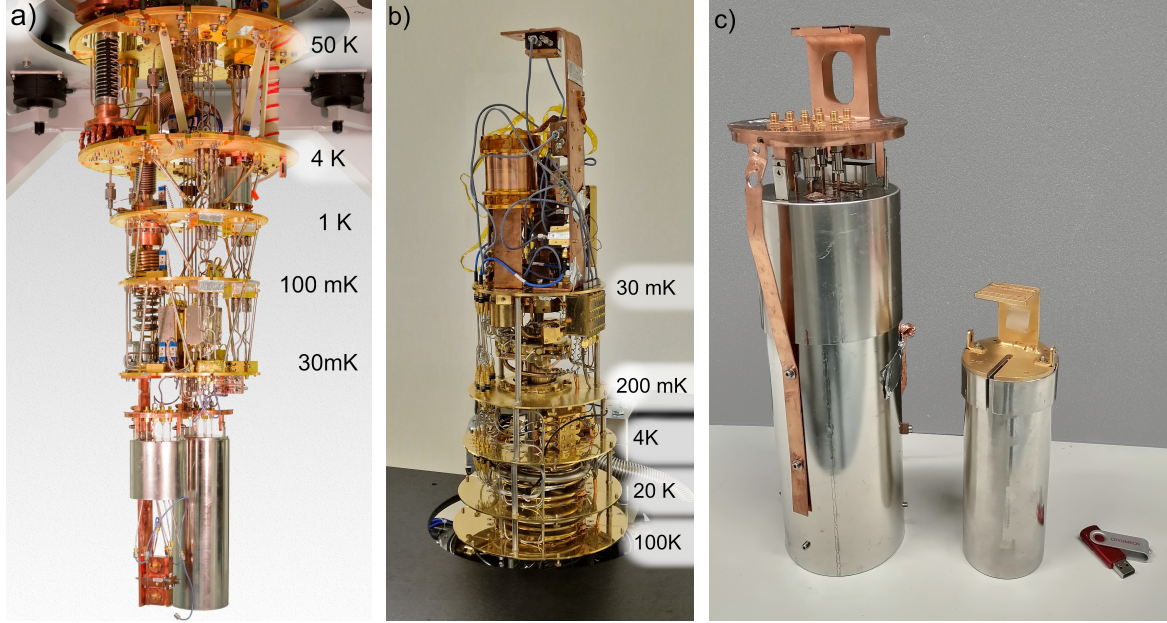


Fig. 5.1 Pictures of the cryostats and shielding cans. (a) Oxford Triton 400. The samples are mounted at the bottom and enclosed by mu-metal shields. (b) Qiniu SL-101-Pro. Here the samples are mounted at the top. (c) Magnetic shields used in different cooldowns: the larger one is a double-layer mu-metal shield, while the smaller one consists of a single mu-metal layer. The larger shield corresponds to the can mounted in picture (a).

Both cryostats consist of multiple temperature stages, each progressively colder towards the sample stage. The first cooling stage is provided by a pulse tube cooler, which cools the 50 K and 4 K stages of the system. The cooling mechanism is based on adiabatic expansion of helium gas. For further details, the interested reader is referred to [37].

To reach temperatures below 4 K, a second cooling cycle is required, thermally anchored to the 4 K stage. This stage operates with a mixture of the helium isotopes ^3He and ^4He . Through evaporative cooling of this mixture, temperatures down to approximately 800 mK can be achieved. At sufficiently low temperatures, the mixture separates into a dilute phase (rich in ^4He with dissolved ^3He) and a concentrated ^3He phase. The continuous transfer of ^3He atoms from the concentrated phase into the dilute phase requires energy and provides the cooling power at millikelvin temperatures. This mechanism enables an endothermic temperature of approximately 20 mK. For a detailed explanation, see [38].

To measure the resonance frequencies of the prototypes, microwave wiring from room tem-

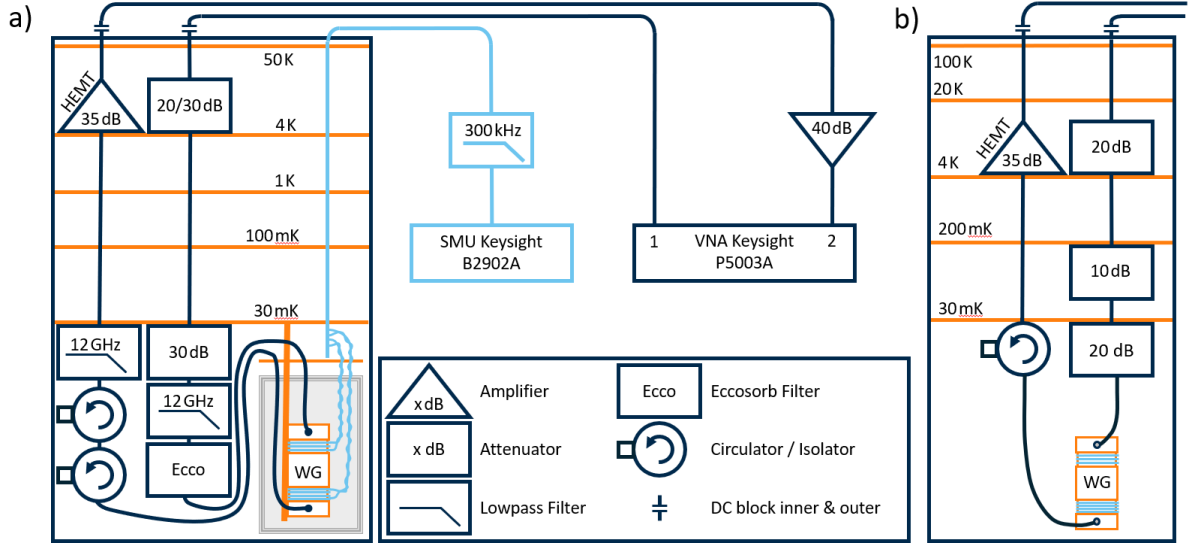


Fig. 5.2 Schematic illustration of the microwave wiring used in the experiment. (a) Wiring of the Oxford fridge with microwave and DC source. (b) Microwave wiring of the Qinu fridge. Figure adapted from [39].

perature to the sample is required. The wiring configuration of the Oxford cryostat is shown schematically in Fig. 5.2. The wiring of the Qinu system is conceptually similar. For microwave excitation and readout we used vector network analysers (VNAs), specifically either the Keysight P5003A or the Rohde & Schwarz ZNB. The input signal is attenuated inside the cryostat at the 4 K and 30 mK stages in order to suppress thermal noise originating from room temperature. Additionally, two 12 GHz low-pass filters are installed to limit the noise frequency bandwidth. After reaching the waveguide containing the sample, the signal passes through a double isolator, each with an isolation of 20 dB. This prevents noise or reflections from the output line from propagating back towards the device. The transmitted signal is first amplified by a high electron mobility transistor (HEMT) amplifier at 4 K and subsequently amplified again at room temperature.

Additional direct-current (DC) lines are required to generate magnetic fields at the sample position. A source measure unit (SMU) was used as a current source to precisely control the current through the field coils. The DC wires are pairwise twisted in order to minimise the effective loop area and thereby reduce magnetic pickup noise from the environment. Below the 4 K stage, superconducting wiring is used to avoid resistive losses. Any dissipation in these lines would locally heat the sample stage and increase the load on the dilution cryostat.

5.1.2 Sample mounting and DC coils

The samples are mounted inside a rectangular waveguide, as shown in Fig. 5.3. It is possible to mount up to 4 sample chips by inserting them from the top and clamping them in place. The input microwave signal enters through the coupling pin, visible as the small disc in Fig. 5.3(b) and propagates along the waveguide. It passes the samples mounted inside before reaching the second coupling pin on the other side connected to the output line. This way, the sample is capacitively side-coupled to the transmission line, commonly referred to as the hanger configuration [40]. The electric field has the strongest amplitude where the sample is mounted in the center of the

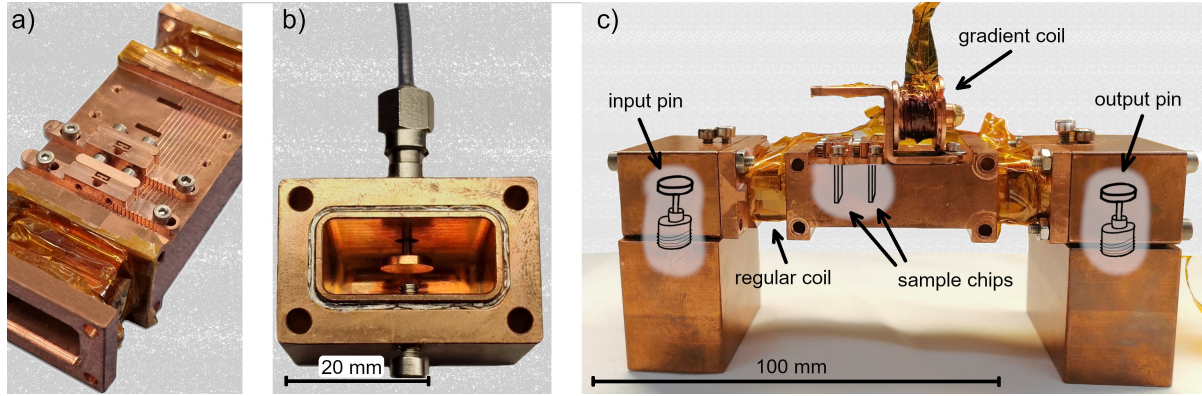


Fig. 5.3 Pictures of the waveguide. (a) Top view of the waveguide without end caps. The samples are clamped in place using a sample holder mounted from above. (b) End cap with the microwave coupling pin. (c) Side view of the fully assembled waveguide with an additional coil mounted on top. The coil wires are covered with orange tape for mechanical protection.

waveguide. The field points up and down along the short side of the waveguide and thus directly couples to the antenna pads oriented along this axis. The internal cross-sectional dimensions of the waveguide are $23 \text{ mm} \times 11 \text{ mm}$, corresponding to a cutoff frequency of approximately 6.5 GHz. Microwave frequencies above the cutoff propagate with low attenuation, while frequencies below cutoff are exponentially suppressed. For more details on the waveguide, the interested reader is referred to [41].

As shown in Fig. 5.3, three different coils can be mounted on the waveguide. The coils wrapped around the waveguide generate a magnetic field that is approximately homogeneous in the centre of the waveguide cross-section. This coil is used to sweep the magnetic field and determine the offset field. The additional coil mounted on the side of the waveguide generates a deliberately inhomogeneous magnetic field. It was installed in only one cooldown to investigate the influence of field gradients on the device response.

Different mu-metal shields were used to protect the samples from environmental magnetic-field noise. For the first prototype, shielding inside the cryostat was implemented using a double-layer mu-metal can with an outer diameter of 101 mm. For the second prototype, a smaller single-layer mu-metal shield with an outer diameter of 67 mm was used. Both shields are shown in Fig. 5.1(c). In one cooldown of the second prototype, the inner shield was removed and only the large external shield surrounding the cryostat was used. Details of all shields used can be found in Table 5.1.

5.2 Characterisation

In this section, the characteristics of the fabricated devices are presented. Different aspects are discussed step by step, including a comparison between the first and second prototype wherever possible.

For the first prototype P1, three nominally identical devices were fabricated, differing only in

Tab. 5.1 Magnetic shields used for the samples. The estimated attenuation assumes a relative magnetic permeability of $\mu_r = 10^4$ and is calculated for an ideal, infinitely long cylindrical shield without openings, using formulas from [42]. In practice, openings and sharp edges reduce the achievable attenuation.

	Small single-layer	Double-layer	Triple-layer	Outside
Outer diameter (mm)	67	101	101	500
Thickness (mm)	1	1	1	2
Material	Mu-metal	2× Mu-metal	Mu-metal – Nb – Mu-metal	Mu-metal
Estimated shielding factor	30	460	>460	8

oxidation dose and therefore in junction inductance. Across five cooldowns, the lowest-frequency device was measured in all five, the medium-frequency device in several cooldowns, and the highest-frequency device only once. In each cooldown, extensive flux maps were recorded to characterise the devices independently. The three versions are sufficiently similar in all their relevant properties that, in the following discussion we will not refer to a specific one, but use P1 for any of them.

Two versions of the second prototype P2 were fabricated, with different frequencies as well as slightly different area ratios, hereafter referred to as P2.1 and P2.2. They were cooled down together and measured across three different cooldowns.

5.2.1 Flux maps

By flux map, we refer to a transmission measurement in which two external parameters are swept: the probe frequency and the magnetic flux through the SQUID loops.

Let us first take a closer look at the frequency sweep at a fixed flux point. The sample is mounted in the hanger configuration, where it is capacitively side-coupled to the transmission line, as mentioned earlier in Sec. 5.1.2 [40]. For a cavity with frequency ω_c and a probe tone with frequency ω_p , the transmission can be described by the complex scattering parameter [43, 44]

$$S_{21}(\omega_p) = a e^{i\alpha} e^{-i\omega_p \tau} \left(1 - \frac{(Q_{\text{tot}}/|Q_{\text{ext}}|) e^{i\varphi}}{1 + 2iQ_{\text{tot}}(\omega_p/\omega_c - 1)} \right). \quad (5.1)$$

Here, the first term accounts for the environment and describes the signal with attenuation a , phase offset α , and delay τ caused by the waveguide itself and all other microwave wiring. This is the part that can be removed from the measured signal by performing a background subtraction. The second term in brackets describes the scattering of the signal at the cavity; the resonance appears as a dip in the transmitted signal. Its shape depends on the total quality factor Q_{tot} and the external (or coupling) quality factor $|Q_{\text{ext}}|e^{i\varphi}$. The latter can be complex to account for an impedance mismatch φ between input and output line. More details on the quality factors are discussed in Sec. 5.2.2. When the real part of the scattering parameter is plotted against the imaginary part, the resonance traces out a circle, hereafter referred to simply as the “Q-circle”.

An example of this is shown in Fig. 5.10; a more extensive discussion and visualisation can be found in [39].

The second axis of the flux map, the magnetic flux, is controlled by applying a DC current through coils near the sample, thereby generating magnetic flux through the loops. The two-dimensional transmission data allow us to extract the resonance frequency as a function of magnetic field. The characterisation of the devices is performed by fitting the model function discussed in Sec. 3.2.3 to the extracted resonance frequencies.

First Prototype

In Fig. 5.4(a), the flux map of the first prototype is shown as a 2D colour map for a current and frequency sweep. The transmission signal is background-corrected, i.e. the contribution of all non-device microwave components is subtracted from the measured signal. In this way, the resonance frequency of the prototype appears as a clear dip in the transmission.

Most flux maps used for characterisation span a wide range in current and frequency. Therefore, the frequency sampling rate was typically chosen to be approximately 1 MHz. This allowed acquisition of the data in a reasonable time while still providing several data points within each resonance dip. From the flux map, the resonance frequency is extracted, as shown in Fig. 5.4(b) along with fits to the model derived in Sec. 3.2.1.

The resonance frequency at each flux value was estimated by selecting the frequency corresponding to the minimum transmission amplitude. A full circle fit, which also considers the phase response, could in principle improve the accuracy of the resonance frequency determination, but was not used because the minimum-transmission approach proved more robust. In particular, the low frequency resolution (only a few data points within the resonance dip) and resonance dips with irregular shapes due to readout power effects and two-level crossings made it difficult to apply a circle-fit routine reproducibly. More details on this are discussed in Sec. 5.2.2.

Second Prototype

The corresponding measurement for the second prototype is shown in Fig. 5.4(c), where both devices P2.1 and P2.2 can be seen. The different magnetic-field periods do not arise from differences in loop area, which are nearly identical, but from the different distances to the coil generating the magnetic field. From the transmission data, it is immediately apparent that the signal-to-noise ratio of the second prototype is significantly improved compared to the first one. Furthermore, the resonances are clearly broader. The combination of these two effects allowed much faster acquisition of the flux maps, since less averaging was required and the frequency sampling rate could be increased without compromising resonance identification.

The resonance frequencies were again extracted by selecting the frequency corresponding to the minimum transmission amplitude and are plotted together with the fit function in Fig. 5.4(d), using the model derived in Sec. 3.2.2.

From the fit function, the parameters of the two fabricated samples of the second prototype were extracted. The results are summarised in Table 5.2. In Fig. 5.5, the circuit from Sec. 3.2.2, including the fitting variables, is shown again. The offset-field parameter is also part of the

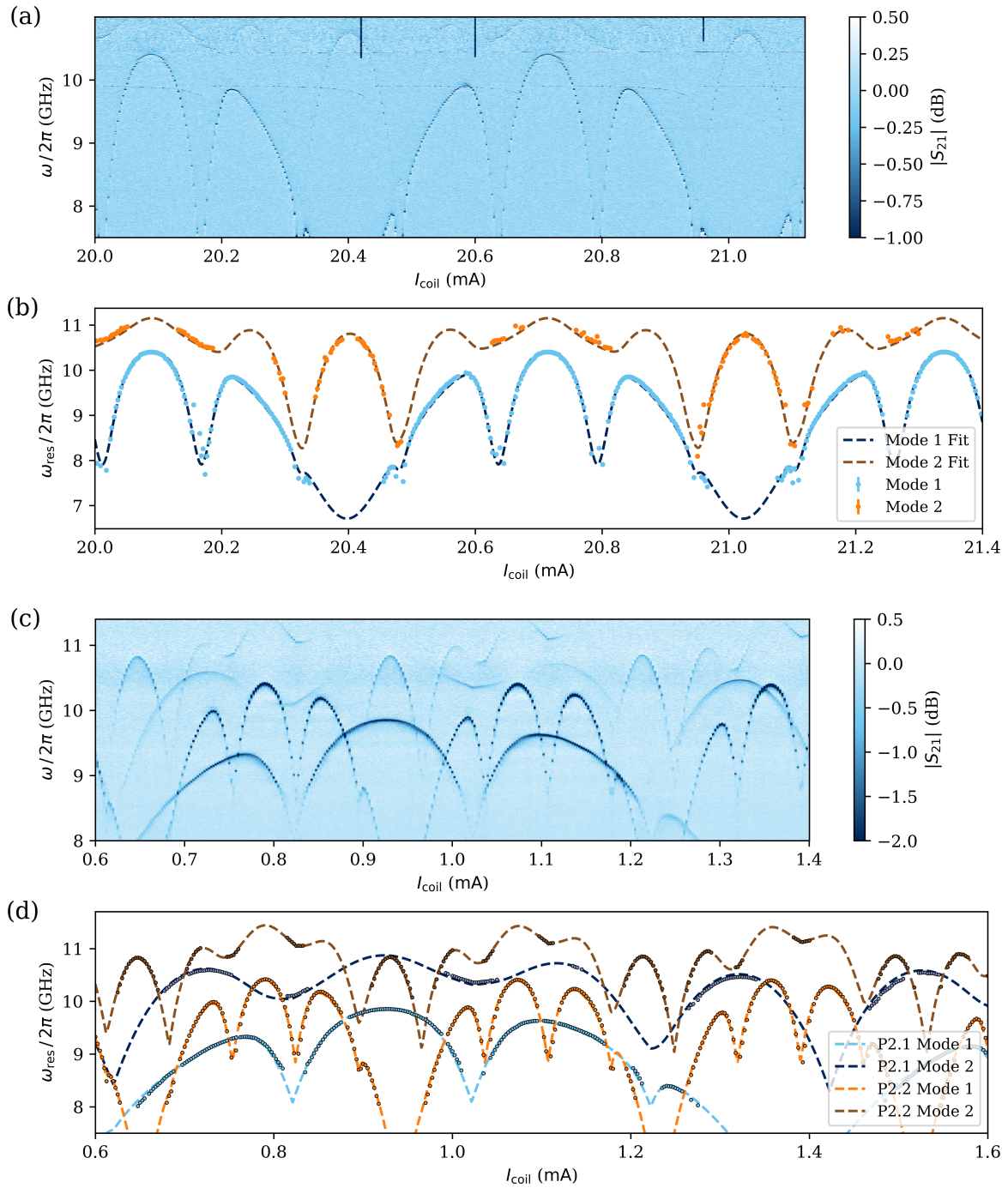


Fig. 5.4

Flux maps and analytic fits for the first and second prototype. (a) Transmission $|S_{21}|$ of the first prototype measured as a function of coil current and frequency; the strongly tuning sample acts as the magnetic-field sensor and both hybridised modes are visible. The flat, nearly horizontal lines originate from other devices mounted in the waveguide and are not considered in this work. The prominent vertical features are measurement artefacts. (b) Extracted resonance frequencies after removing spurious modes; analytic fit to the flux map. (c) Same as (a), but measured for the second prototype. P2.1 and P2.2 with slightly different parameters are visible, one positioned closer to the field-generating coil. (d) Same as (b), but for the second prototype.

fit but is discussed separately. The fit function has 13 free parameters and tends to become trapped in local minima, failing to converge to the global minimum when the initial guess is too far from the true solution. To improve convergence, we fitted the data of both prototypes simultaneously. Parameters expected to be identical for both devices were constrained to be equal, thereby reducing the number of independent degrees of freedom. This resulted in a total of 18 fitting parameters for the joint fit, corresponding to effectively 9 free parameters per prototype.

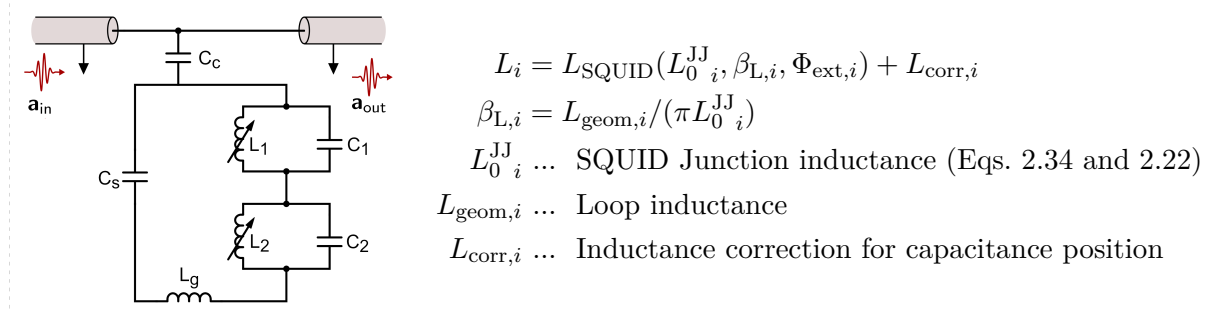


Fig. 5.5 Circuit model used in the fit function. The inductance L_i in the model depends on the external flux $\Phi_{\text{ext},i}$. Further details on the parameters are discussed in Sec. 3.2.2.

Tab. 5.2 Fit parameters of the double-SQUID model for the second prototype. Shared parameters are identical for P2.1 and P2.2; device-specific parameters are listed separately.

Parameter	Estimated	Estimated by	P2.1	P2.2
C_1 (fF)	202	HFSS simulation	$4.5(2) \cdot 10^2$	
$L_{\text{geom},1}$ (pH)	452	geometry + Eq.5.2	$4.5(2) \cdot 10^2$	
$L_{\text{corr},1}$ (pH)	—	—	$-51(3)$	
$L_{0,1}^{\text{JJ}}$ (pH)	1100	resistance + Eq.4.2	$8.2(4) \cdot 10^2$	$7.3(4) \cdot 10^2$
$\beta_{L,1}$	0.13	Eq. 2.43	0.172(7)	0.198(8)
C_2 (fF)	186.2	HFSS simulation	$4.0(2) \cdot 10^2$	
$L_{\text{geom},2}$ (pH)	2930	geometry + Eq.5.2	$2.93(15) \cdot 10^3$	
$L_{\text{corr},2}$ (pH)	—	—	$-8.8(4) \cdot 10^2$	
$L_{0,2}^{\text{JJ}}$ (pH)	1400	resistance + Eq.4.2	$1.34(7) \cdot 10^3$	$1.26(6) \cdot 10^3$
$\beta_{L,2}$	0.67	Eq. 2.43	0.695 (4)	0.742 (4)
$C_s + C_c$ (fF)	—	—	43(2)	
L_g (pH)	340	geometry + Eq.5.2	$2.1(1) \cdot 10^{-2}$	
Area ratio	—	—	4.151(2)	4.092(2)
I_P ($\mu\text{A}/\Phi_0$)	—	—	200.89(15)	70.78(12)

The fitted capacitance and inductance parameters are strongly correlated. Even though the model contains many parameters, the situation is similar to that of a simple LC resonator with a known resonance frequency: any inductance–capacitance combination with the same product yields the same resonance frequency. Therefore, at least one parameter must be fixed in order to extract meaningful values for the others.

In our case, the geometric loop inductances were fixed. They should be nearly identical for P2.1

and P2.2, and their values can be estimated reliably from the loop circumference and conductor cross-section. We used the formula

$$L_0 = 2 \cdot 10^{-7} l \left(\ln \left(\frac{2l}{w+t} \right) + 0.5 + 0.2235 \frac{w+t}{l} \right) \quad (5.2)$$

which describes the inductance L_0 of a straight rectangular conductor with length l , width w , and thickness t and is taken from [45]. This formula is valid for low frequencies; however, the deviation at higher frequencies is less than 2 % for length-to-width ratios comparable to those in our case.

As seen in Table 5.2, the extracted junction inductances are slightly lower than expected, resulting in screening parameters β_L that are slightly above the design values.

The parameter deviating most strongly from expectations is L_g . The best fit is obtained when this parameter approaches zero. It is likely not reduced further because the numerical evaluation becomes unstable for even smaller values. This is noteworthy, since the circuit model shown in Fig. 3.7 suggests that this inductance should be significant. Setting this parameter to zero would simplify the model and allow the use of the reduced impedance expression with two instead of three solutions.

The fitted capacitance values are approximately twice as large as predicted by HFSS simulations. One possible explanation is that the circuit simplification shown in Fig. 3.6 is not fully justified. For example, inductive contributions in the capacitive branch are neglected in the model and could modify the effective capacitance contributing to the modes.

The fitting parameter I_P in $\mu\text{A}/\Phi_0$ relates the coil current to the magnetic flux through the loop. Knowing the area of the SQUID loops allows us to convert the coil current into an effective magnetic field by using Eq. 3.27.

5.2.2 Quality Factors

In this section, we analyse the quality factor Q , which characterises the ratio of energy loss in an oscillating system. It is defined as

$$Q = 2\pi \cdot \frac{\text{energy stored}}{\text{energy dissipated per cycle}} \xrightarrow{\text{LC resonator}} Q = \frac{\omega_{\text{res}}}{\kappa}. \quad (5.3)$$

In case of an LC resonator, it determines how sharp a resonance is, where κ is the linewidth of the resonance dip. We distinguish between the internal quality factor Q_{int} and the external quality factor Q_{ext} . The internal quality factor is related to the internal linewidth κ_{int} and characterises intrinsic losses of the device. The external quality factor is associated with the external coupling between the drive and the device and therefore with the external linewidth κ_{ext} . Together they combine into the total quality factor

$$Q_{\text{tot}} = \frac{1}{1/Q_{\text{ext}} + 1/Q_{\text{int}}}. \quad (5.4)$$

The total quality factor determines the linewidth of the resonance. Equally important is the ratio between Q_{int} and Q_{ext} . How the transmission signal behaves is described by Eq. 5.1. We can classify the behavior into three coupling regimes.

For $Q_{\text{int}} > Q_{\text{ext}}$ the system is overcoupled. In the weak-driving regime (no saturation effects, as discussed later in Sec. 5.2.3), strong overcoupling leads to resonance dips several tens of dB deep in the transmission. The resonance appears deep, broad, and clearly visible.

In the critically coupled case $Q_{\text{int}} \approx Q_{\text{ext}}$, the resonator dissipates and couples comparable amounts of energy. The resonance is exactly 3 dB deep. For fixed Q_{int} , the resonance becomes narrower as Q_{ext} increases.

For $Q_{\text{int}} < Q_{\text{ext}}$ the system is undercoupled with a transmission dip smaller than 3 dB at resonance. As undercoupling increases, the resonance dip becomes shallower and can eventually disappear into the noise floor.

Looking at the transmission measurements of both prototypes in Fig. 5.4(a,c), one can see that the external coupling of the modes changes significantly across the flux map. In the hybridised regime, the symmetric (lower-frequency) mode couples strongly to the waveguide, while the anti-symmetric mode does not couple at all. The reason lies in the mode structure shown in Fig. 3.5: the antisymmetric mode produces no net charge dipole on the antenna pads, or equivalently no charge on the coupling capacitance in the circuit model. Consequently, this mode is effectively decoupled from the measurement line.

The upper mode is generally undercoupled and becomes visible only when the SQUID eigenfrequencies are far detuned. The lower mode, in contrast, couples most strongly in the hybridised regime (upper turning point).

First Prototype

The first prototype exhibits significantly sharper resonances than the second. A closer inspection of the flux map reveals numerous avoided crossings with "dark" modes, as shown in Fig. 5.6.

These additional modes couple to the waveguide only through the sensor device, indicating that they originate in close proximity to it. A plausible explanation is the presence of two-level systems (TLS) in the junction oxide, which couple to the electric field. Compared to typical qubits, the density of TLS may be enhanced due to the long oxidation times used in this prototype. Moreover, since the junctions provide the dominant capacitance, the electric field is concentrated in the oxide barrier, leading to strong coupling to TLS defects.

The total quality factor of the first device was estimated to be

$$Q_{\text{tot}} = 3.5 \times 10^3,$$

evaluated at the sweet spot (upper turning point) of the lower mode at a position without obvious avoided crossings with two level systems. The corresponding internal and external quality factor were

$$Q_{\text{int}} = 7.2 \times 10^3, \quad Q_{\text{ext}} = 6.9 \times 10^3.$$

At flux points on steeper slopes of the flux map, we measured

$$Q_{\text{tot}} = 1.4 \times 10^3,$$

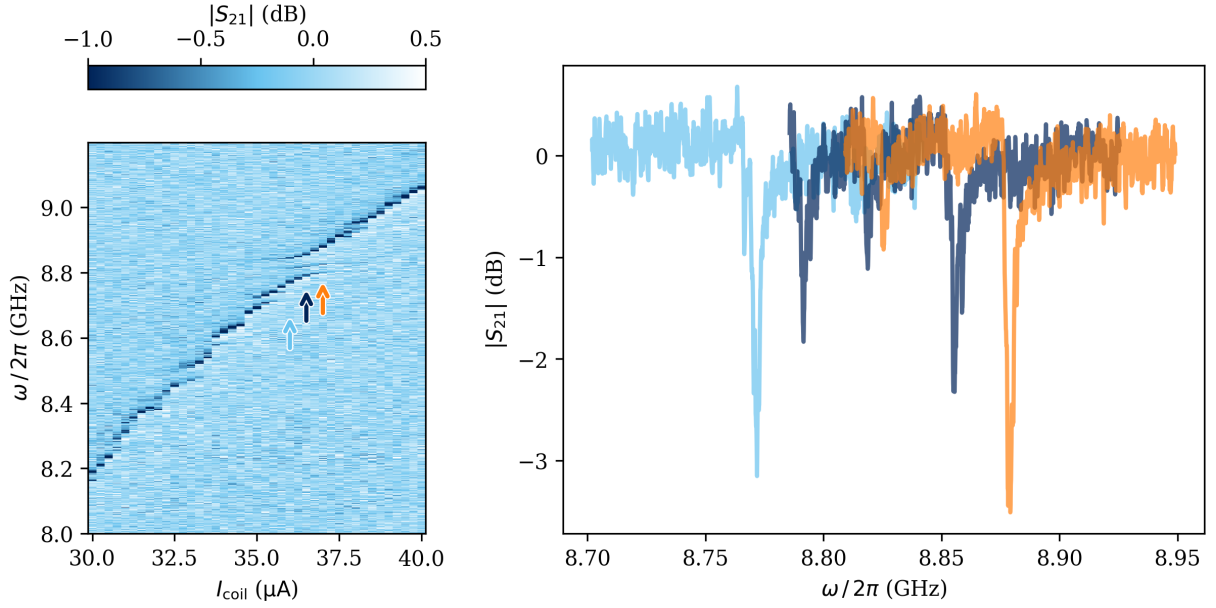


Fig. 5.6 Zoom into the flux map of the first prototype. Left: Several avoided crossings with "dark" modes are visible. Arrows indicate the slices plotted on the right. Right: Three slices of the flux map. Individual slices can exhibit multiple dips due to avoided crossings, making it difficult to determine the exact resonance frequency.

in a strongly undercoupled regime. This reduction is caused by two effects: an increase in Q_{ext} due to weaker coupling of the mode to the waveguide, and a decrease in Q_{int} due to flux noise. Flux noise effectively broadens the resonance, especially in regions with large frequency slope versus flux, thereby reducing the effective internal quality factor. In addition, there are strong variations in the quality factor, due to residual coupling to nearby modes as shown in Fig. 5.6.

Second Prototype

The second prototype exhibits significantly broader resonances and therefore lower quality factors. At the sweet spot (upper turning point, no flux-noise broadening) of the lower mode, we obtained

$$Q_{\text{tot}} = 230,$$

with

$$Q_{\text{int}} = 507, \quad Q_{\text{ext}} = 420.$$

This corresponds to nearly critical coupling. The large coupling ratio for P2 improves mode visibility and allows faster measurements with fewer frequency points. Avoided crossings with TLS were not observed for this sample. However, the internal quality factor is considerably lower than for the first prototype.

One possible explanation is fabrication-related contamination, as the sample had to leave the cleanroom during processing. In addition, the measurement was performed with relatively weak magnetic shielding and limited DC filtering, leading to enhanced flux noise.

It is apparent from Fig. 5.4 that Q_{int} drops substantially in regions with steep flux dependence, indicating that flux-noise has a significant contribution to the effective internal quality factor at these points. Even in the nominally flat region, it is likely that a small residual slope remained, contributing to additional broadening.

5.2.3 Power behaviour

In this section, we investigate the behaviour of the system as a function of readout power. From this, we attempt to estimate the anharmonicity of the device. All measurements shown here were performed on the lower mode, as this mode couples more strongly to the waveguide. Experiments on the upper mode did not yield usable results, since it is already undercoupled and increasing the input power further reduced the signal-to-noise ratio.

Before looking at the data, let us consider the expected power dependence. The power behaviour of an anharmonic resonator depends on the ratio of the anharmonicity, expressed by the Kerr constant $\mathcal{K}$ from Eq. 2.30, to the transition linewidth κ_{tot} . Two well-known regimes with different characteristics can be distinguished:

- **Transmon regime $\mathcal{K} \gg \kappa_{\text{tot}}$**
In this regime, the linewidth is much smaller than the anharmonicity of the resonator. Consequently, an incoming probe tone at a fixed frequency predominantly drives the $|0\rangle \rightarrow |1\rangle$ transition. The transition frequency of the next higher transition $|1\rangle \rightarrow |2\rangle$ is sufficiently detuned for low powers such that no significant excitation occurs, and the system behaves effectively as a two-level system. At single-photon input power, this transition behaves similarly to that of a harmonic resonator. However, increasing the drive power saturates the $|0\rangle \rightarrow |1\rangle$ transition, thereby limiting the photon scattering rate. A further increase in input power decreases the ratio of scattered to transmitted photons, resulting in a reduced dip depth at resonance. A modified model for the scattering parameter in this regime can be found in [10]. For such a system, two-tone spectroscopy can be used to measure the anharmonicity. This is achieved by placing a strong probe tone at the $|0\rangle \rightarrow |1\rangle$ transition to saturate it, while simultaneously applying a weak probe tone to detect the $|1\rangle \rightarrow |2\rangle$ transition.
- **Weakly anharmonic Kerr resonator $\mathcal{K} \ll \kappa_{\text{tot}}$**
In this regime, the linewidth is much larger than the anharmonicity of the resonator. An incoming probe tone drives not only the $|0\rangle \rightarrow |1\rangle$ transition but also higher transitions. At single-photon input power, the system again behaves similarly to a harmonic resonator. At higher drive powers, the cavity photon number $\bar{n}_c$ increases, leading to a shift of the resonance frequency according to Eq. 2.31. In this case, the amplitude of the resonance position shifts and the shape resembles that of a Duffing oscillator [46]. For moderate drive powers, the cavity photon number $\bar{n}_c$ is proportional to the incoming photon rate $\bar{n}_{\text{in}}$. Therefore, the Kerr constant $\mathcal{K}$ can be extracted by relating the resonance frequency shift to $\bar{n}_c$.

At sufficiently high input powers, the system enters a bistable regime, where two stable states with different photon numbers coexist for a single drive tone. In this regime, the cavity response exhibits hysteresis: the measured response depends on the sweep direction

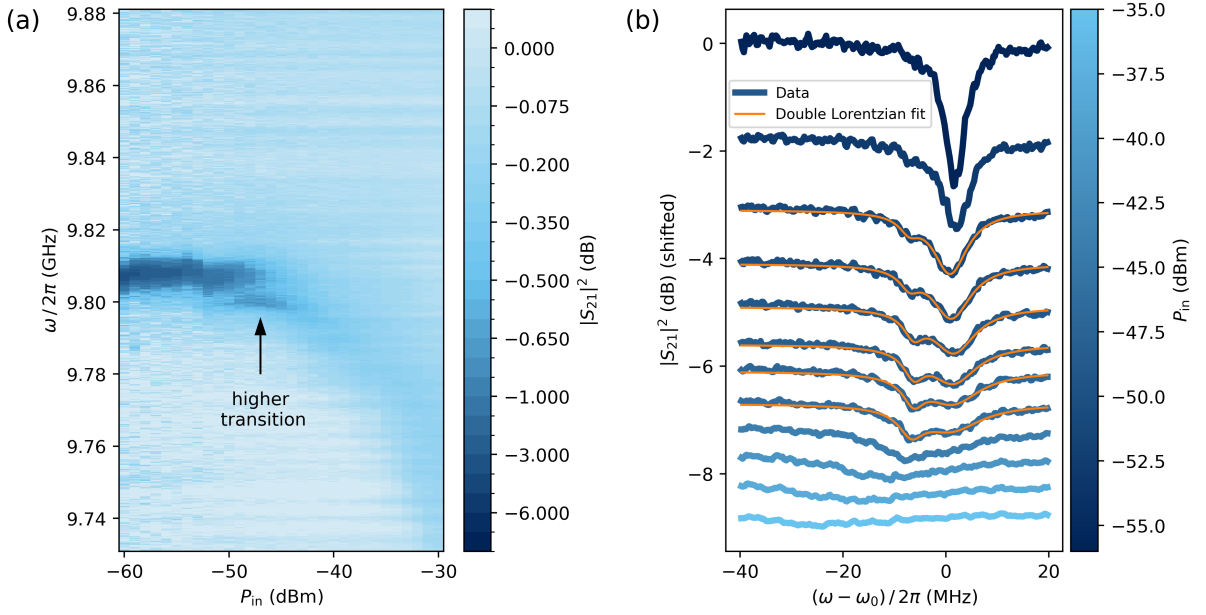


Fig. 5.7 Power-sweep behaviour of the lower mode of the first prototype. (a) Colour map of $|S_{21}|$ for different input powers. With increasing power, the resonance shifts to lower frequencies. (b) Selected slices from (a). In a certain power range, a second peak becomes visible. The data were fitted with a double Lorentzian to estimate the anharmonicity.

(from high to low frequencies or vice versa). Transitions between the states may also occur due to fluctuations. More details on this behaviour can be found in [39, 41].

First Prototype

In Fig. 5.7(a), the transmission response of P1 for different input powers is shown. Two effects are visible. First, the resonance frequency shifts to lower values with increasing power, indicating that the anharmonicity is small and comparable to the linewidth. Second, the resonance dip becomes shallower with increasing readout power, which is characteristic of saturation effects. The dip becomes shallower even before a pronounced frequency shift occurs, and for high powers the transition is clearly saturated. This indicates that the device operates in an intermediate regime between a weakly anharmonic Kerr resonator and the transmon regime.

In this case, estimating the anharmonicity by relating input power to frequency shift is no longer straightforward due to transition saturation. Although the frequency shift remains related to $\bar{n}_c$, the relation between the cavity photon number $\bar{n}_c$ and incoming photon rate $\bar{n}_{in}$ is no longer linear, making a direct extraction of the anharmonicity difficult.

For some power sweeps, a direct estimation of the anharmonicity was nevertheless possible. One representative example is shown in Fig. 5.7.

In the colour map and the line cuts, a second peak becomes clearly distinguishable at intermediate powers. This feature could correspond either to the $|1\rangle \rightarrow |2\rangle$ transition or to a two-photon $|0\rangle \rightarrow |2\rangle$ transition into the second excited state. In Ref. [47], a system in the crossover regime

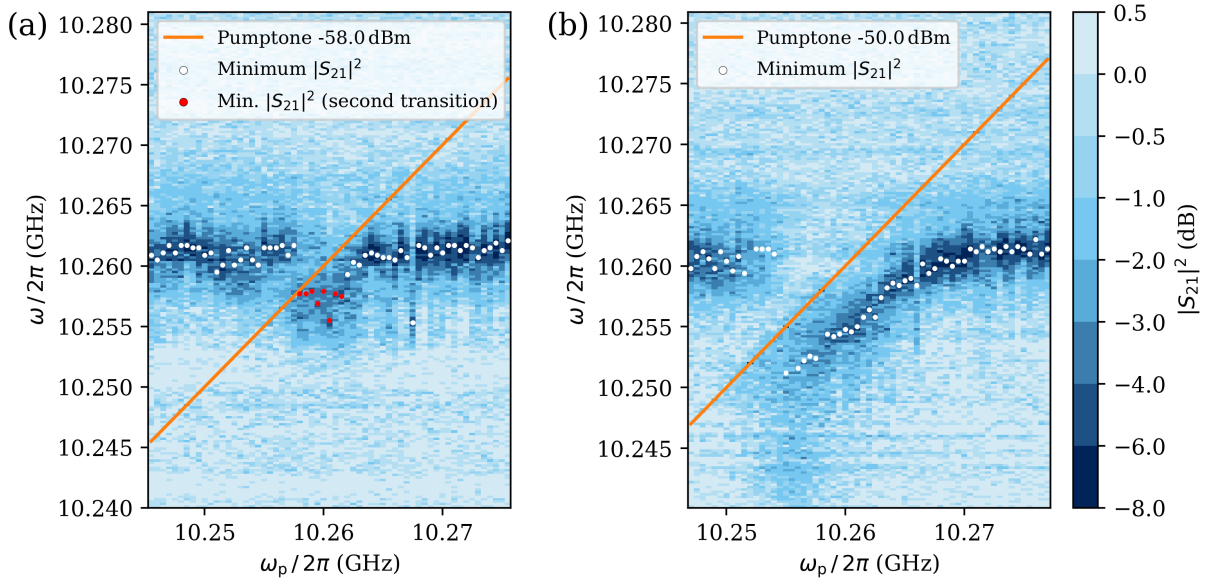


Fig. 5.8 Two-tone measurements on the lower mode. A strong pump tone (orange) at frequency ω_p is combined with a weak VNA probe tone. (a) For suitable pump powers, a discrete frequency shift is observed. (b) For higher pump powers, the discrete features vanish and a continuous frequency shift appears.

between a transmon and a weakly anharmonic Kerr resonator is studied. The authors report spectra very similar to our observations and attribute the second peak to the $|0\rangle \rightarrow |2\rangle$ transition.

Assuming that the same interpretation applies here, the anharmonicity can be estimated using $\omega_{02}/2 = \omega_{01} + \mathcal{K}/2$. In this case, the observed frequency splitting corresponds to half the anharmonicity. From the data, we estimate $\mathcal{K}/2\pi = -15.6(2)$ MHz. At this operating point, the transition linewidth is $\kappa_{\text{tot}}/2\pi = 5.8$ MHz, resulting in a ratio $\mathcal{K}/\kappa_{\text{tot}} \approx 2.7$.

It should be noted that the anharmonicity depends on the junction participation ratio and the capacitive energy, and therefore varies with flux as the SQUID inductance changes. The apparent linewidth of the resonance also changes across the tuning range. Furthermore, this method is only applicable when the linewidth and anharmonicity are of comparable magnitude. For most power sweeps, no clearly distinguishable second peak was observed.

We also performed two-tone measurements on P1. A microwave pump was combined with a weak VNA probe estimate the anharmonicity, as typically done in the transmon regime. The pump power and pump frequency were swept, while the VNA scanned a narrow frequency range around the resonance.

For specific pump powers, a second transition appeared that remained stable over a small range of detunings between the pump tone and the resonance, as indicated in Fig. 5.8(a) as red points. This behaviour is consistent with the $|1\rangle \rightarrow |2\rangle$ transition while the $|1\rangle$ state is populated by the pump tone. From this measurement, an anharmonicity of $\mathcal{K}/2\pi = -3.65(12)$ MHz was extracted at this flux point.

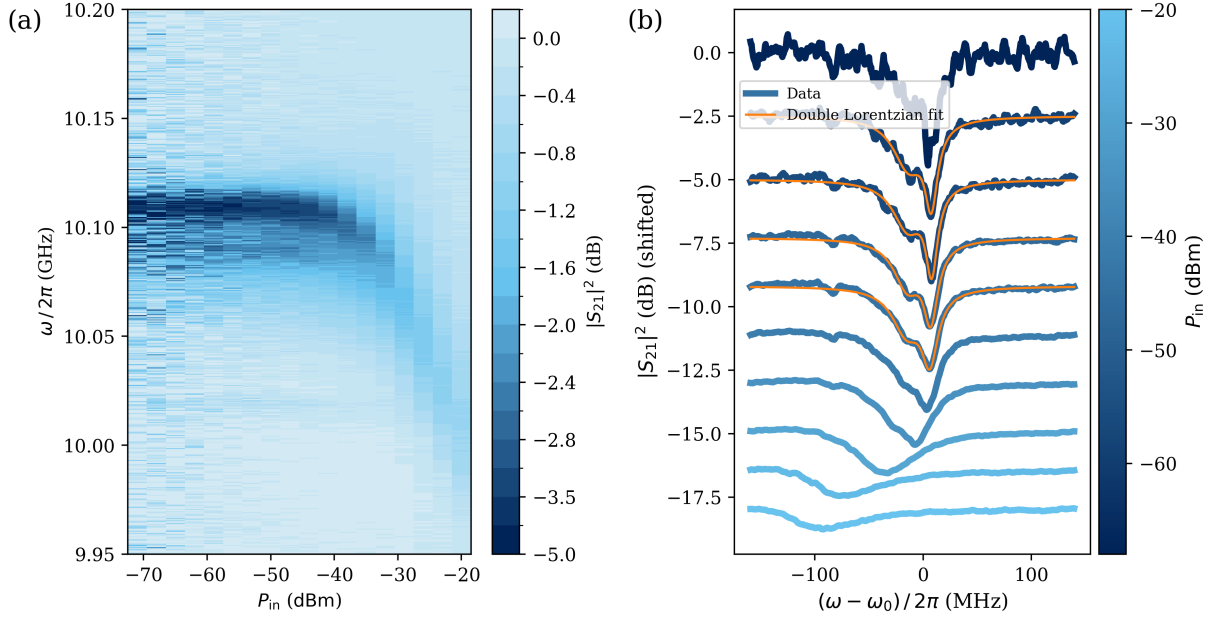


Fig. 5.9 Power-sweep behaviour of the lower mode of the second prototype. (a) Colour map of $|S_{21}|$ versus input power. (b) Selected line cuts. A second peak becomes visible in a certain power range and was fitted with a double Lorentzian to estimate the anharmonicity.

At higher pump powers, no distinct transitions were observed. Instead, the resonance shifted almost continuously, as shown in Fig. 5.8(b). Although discrete transitions might be expected, this behaviour is consistent with results reported in [48]. There, a continuous frequency shift was observed due to cross-Kerr interactions between modes. In our case, partial population of multiple transitions may lead to overlapping spectral features whose weighted superposition produces an effectively continuous shift.

We also reversed the direction of the frequency sweep to investigate possible bistability. In a bistable Kerr resonator, one would expect hysteresis effects depending on whether the frequency is swept from high to low or vice versa. However, no significant differences between the two sweep directions were observed within our measurement resolution.

Second Prototype

The second prototype shows qualitatively similar behaviour but with broader linewidths. The power-dependent frequency shift is shown in Fig. 5.9. The lineshape in this case differs slightly from that of the first prototype. A second peak appears at low input powers and is located very close to the main resonance. We therefore attribute the two peaks to the $|0\rangle \rightarrow |1\rangle$ and $|1\rangle \rightarrow |2\rangle$ transitions.

From a double-Lorentzian fit, an anharmonicity of $\mathcal{K}/2\pi = -21.6(2)$ MHz is obtained. The same fit also yields the linewidths of the two transitions, with $\kappa_{01}/2\pi = 9.2$ MHz for the $|0\rangle \rightarrow |1\rangle$ transition and $\kappa_{12}/2\pi = 16.4$ MHz for the $|1\rangle \rightarrow |2\rangle$ transition. Using these values, we obtain a ratio of $\mathcal{K}/\kappa_{01} \approx 2.3$.

Compared to the first prototype, the anharmonicity is slightly larger. However, due to the increased linewidth, the second prototype operates closer to the regime of a weakly anharmonic resonator.

We can use Eq. 2.32 together with the fitted circuit parameters to estimate the expected anharmonicity. For this, the capacitive energy of the mode is required. As an approximation, we use the expressions of the simplified circuit model discussed in Sec. 3.2.1.

We assume a junction participation ratio of $p_J \approx 0.5$. The participation ratio of the junction will change with the mode structure and the junction inductance (with changing flux) but will stay within the bounds $[0.2, 0.9]$ and could potentially be lower at points high in the flux map. We obtain approximate values of $\mathcal{K}/2\pi \approx 21$ MHz for the normal mode, $\mathcal{K}/2\pi \approx 38$ MHz for the symmetric mode, and $\mathcal{K}/2\pi \approx 11$ MHz for the antisymmetric mode (see Eqs. 3.20, 3.22, 3.23). Since the measurement was performed on the lower hybridised mode, its anharmonicity is expected to lie between the symmetric and normal-mode limits.

Comparing these estimates to the measured value of the second prototype shows good agreement. In contrast, the extracted values for the first prototype are considerably smaller. For this device, we do not have reliable capacitance and inductance values from a global fit. Qualitatively, however, the first prototype contains less linear inductance and smaller junction inductance, resulting in a larger effective capacitance and therefore a smaller charging energy. One would therefore expect the anharmonicity to be reduced by roughly a factor of two compared to the second prototype.

The value obtained from the double-Lorentzian fit of the first prototype, $\mathcal{K}/2\pi = -15.6(2)$ MHz, is consistent with this expectation. However, the value extracted from the two-tone measurement, $\mathcal{K}/2\pi = -3.65(12)$ MHz, appears significantly too small. One possible explanation is that a nearby two-level system was inadvertently probed instead of the true $|1\rangle \rightarrow |2\rangle$ transition. Given the high density of spurious modes observed in this device, such a misidentification cannot be excluded.

In Fig. 5.10, a parametric plot of the S_{21} parameter in the complex plane is shown for different powers. With increasing power, saturation reduces the depth of the resonance dip, leading to a deformation of the Q-circle into an ellipse [10]. In addition, higher-level transitions overlap with the ground-state transition, resulting in asymmetric line shapes and peak shifts.

5.2.4 Hysteresis

One important aspect of the magnetic-field sensor is hysteresis, which is investigated in this section. Both prototypes exhibited similar behaviour in this context; therefore, they are discussed together. As described in Sec. 2.3.2, SQUID loops with large β_L can exhibit multiple stable states corresponding to different circulating loop currents, each representing a local minimum of the potential energy.

When the magnetic field applied to a SQUID-resonator is tuned slowly, the loop current generally remains in the same metastable configuration and changes adiabatically, even if another configuration becomes energetically more favourable. At some point, however, the system may abruptly switch to another branch, either due to an external perturbation or because the local

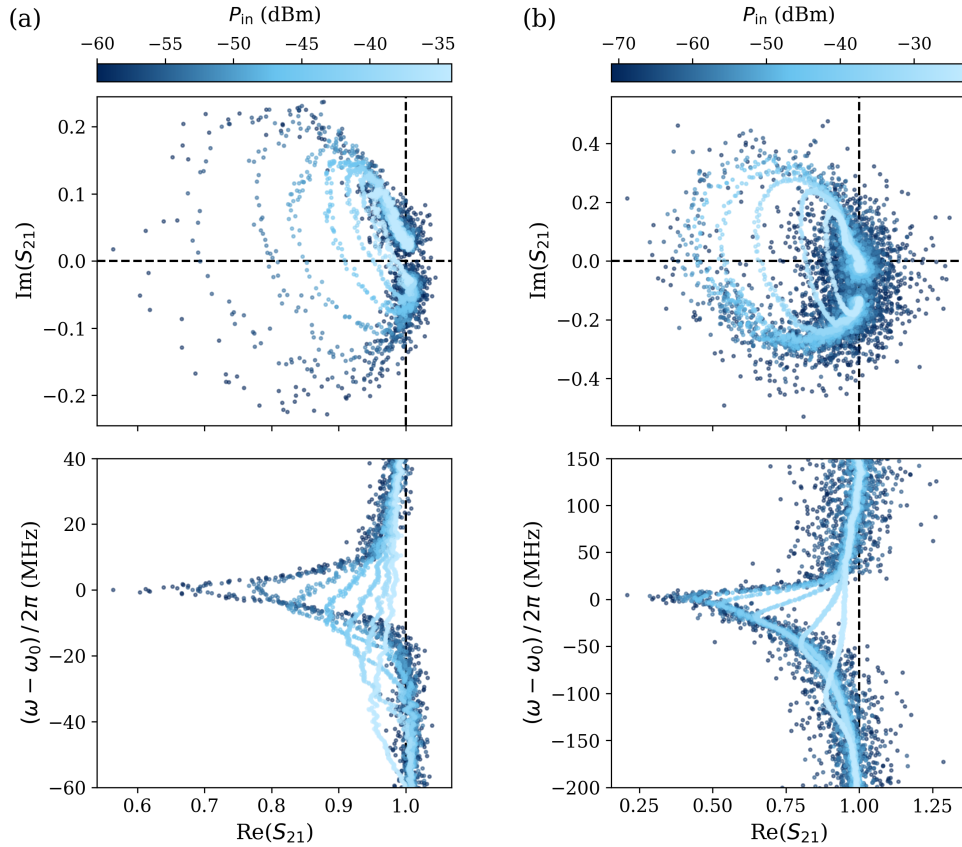


Fig. 5.10 Complex-plane response for different powers. (a) First prototype, (b) second prototype. Bottom: magnitude response versus frequency. Top: complex-plane (Q-circle) representation. Note that the axis scales differ between the two devices.

minimum disappears. Such a transition would manifest as a discontinuous jump in the resonance frequency within the fluxmap. To investigate this effect, we repeatedly measured the response of the sensor over a small magnetic-field range, as shown in Fig. 5.11(a).

The measurement procedure was as follows.

1. The current of the magnetic field generating coil was ramped to a large negative value $-I_{\text{max}}$.
2. Then a fluxmap was recorded while sweeping the current from negative to positive values, duration approximately 30 minutes.
3. Ramping of the coil current to a large positive value $+I_{\text{max}}$.
4. Another flux map with the same range as before was then recorded, but with the flux sweep in the reverse direction (positive to negative).
5. Repeat the process.

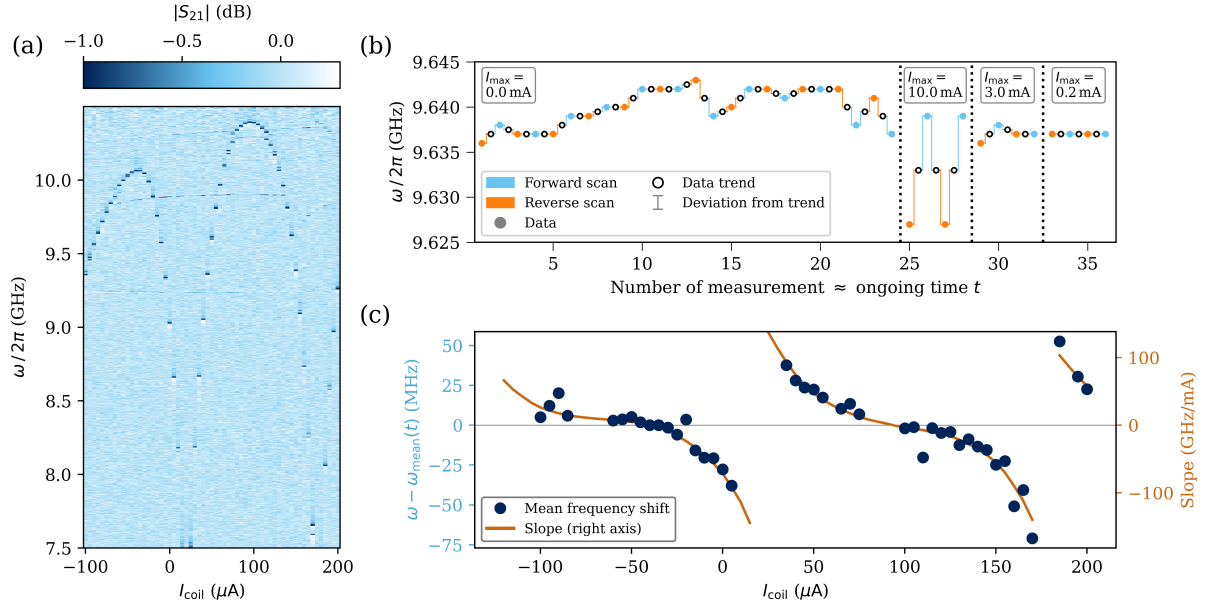


Fig. 5.11 Hysteresis measurement. (a) The same fluxmap section was measured repeatedly to investigate possible hysteresis effects. Other visible samples and resonances around crossing regions were excluded from the analysis. (b) Measured resonance frequency at $I_{\text{coil}} = -95 \mu\text{A}$ for consecutive fluxmaps. The dotted vertical lines separate measurement rows. Black circles represent the mean between two measurements, and the lines indicate the deviation from that mean. The first measurement row was performed with $I_{\text{max}} = 0$ to estimate the frequency shift due to magnetic field fluctuations. The second measurement row with $I_{\text{max}} = 10 \text{ mA}$ clearly shows alternating frequency shifts. (c) Deviation from the mean frequency for the four fluxmaps of the $I_{\text{max}} = 10 \text{ mA}$ row. The slope extracted from the fluxmap is shown on a second axis. Both quantities correlate as expected.

This sequence was repeated, resulting in a total of 36 fluxmaps. Comparing the fluxmaps revealed small frequency differences at identical current-bias values. These deviations, however, were not large enough to clearly indicate flux hysteresis in the resonance frequency itself. Since our measurement window is limited to frequencies above the waveguide cutoff ($\omega/2\pi > 7.5 \text{ GHz}$), hysteretic behaviour at lower frequencies cannot be excluded.

Although no clear frequency hysteresis was observed within the accessible window, the data can be used to estimate magnetic-field hysteresis. This analysis was motivated by long-term measurements (see Fig. 5.15), which indicate that the measured offset field depends not only on the instantaneous coil current but also on previously applied currents.

To quantify this effect, we analysed the frequency difference between consecutive fluxmaps at identical flux points, as illustrated in Fig. 5.11(b). Between each fluxmap, the coil current was ramped alternately to large positive and negative values. Random fluctuations of the external field are expected to average out over many repetitions. In contrast, a hysteresis effect would lead to a systematic alternating shift between consecutive measurements. The first measurement

Tab. 5.3 Maximal current/field and extracted hysteresis width for different cooldowns. The values $I_{\max}$ and $B_{\max}$ were applied briefly before the field measurement, alternately in the positive and negative direction. B_{hyst} is extracted as the average alternating shift. The first two entries do not show a statistically significant deviation of B_{hyst} from zero.

cooldown	$I_{\max}$ (mA)	$B_{\max}$ (μT)	B_{hyst} (pT)	$B_{\text{hyst}}/B_{\max}$
1	0.0	0.00	5.9(21)	–
1	0.2	0.07	2.5(20)	$3.6(29) \cdot 10^{-5}$
1	3.0	1.02	21.9(22)	$2.1(2) \cdot 10^{-5}$
1	10.0	3.41	289(5)	$8.47(15) \cdot 10^{-5}$
2	10.0	0.77	13.0(12)	$1.69(14) \cdot 10^{-5}$
2	30.0	2.31	38.9(12)	$1.68(5) \cdot 10^{-5}$

row was performed with $I_{\max} = 0$ to estimate the effect of magnetic field fluctuations.

Four measurement rows with different values of $I_{\max}$ were analysed. For each flux point, the observed frequency shift was divided by the local slope of the fluxmap in order to convert the frequency deviation into an equivalent shift of the X-axis, which is the coil current. As shown in Fig. 5.11(c), the frequency shift correlates strongly with the local slope, as expected for a field-induced offset. Furthermore, we can relate both this hysteretic current shift as well as $I_{\max}$ to a magnetic field using Eq. 3.27 with the fitting parameter of the calibration and the measured area of the SQUID.

Four measurement rows with different values of $I_{\max}$ were analysed. For each flux point, the frequency difference between consecutive fluxmaps was evaluated. We then used the local slope of the fluxmap, to convert the frequency shift to an equivalent shift of coil current I_{coil} .

As shown in Fig. 5.11(c), the frequency shift scales with the local slope, as expected for an effective shift of the magnetic field. The resulting current shift, as well as $I_{\max}$, can be converted into a magnetic field using Eq. 3.27, together with the calibration factor and the measured SQUID area.

An additional measurement was performed in a separate cooldown for the second prototype. All extracted results are summarised in Table 5.3. It is clearly visible that the hysteresis field B_{hyst} increases with increasing $I_{\max}$. While the data set is not sufficient to determine the exact functional dependence, the ratio is consistently on the order of

$$\frac{B_{\text{hyst}}}{B_{\max}} \approx 10^{-5}.$$

Although this effect is small, it indicates that magnetisation of materials surrounding the waveguide contributes substantially to the effective offset field.

A plausible origin of this behaviour is remanent magnetisation of nearby ferromagnetic components, such as screws, connectors, or parts of the shielding. Even materials that are nominally

non-magnetic can exhibit weak hysteretic behaviour after exposure to relatively large magnetic fields.

5.3 Magnetic field measurements

In this section, we come to the main purpose of the device: magnetic field measurements. We made measurements of the offset field at the device position and also investigated field changes over time. We also studied the change in offset field upon introducing a magnet outside the cryostat.

5.3.1 Offset field for different sample shielding configurations

We attempted to measure the offset field inside the shielding cans for different magnetic-shielding configurations in 10 cooldowns. Unfortunately, in 8 out of these 10 cooldowns we were not able to obtain a conclusive offset field. Simply put, the reason for this was insufficient magnetic shielding. In Table 5.4 the properties relevant for offset field measurements of the two prototypes are shown. For the second prototype we had two fabrication rounds with slightly different parameters.

Tab. 5.4 Comparison of geometrically measured and fitted parameters for the investigated prototypes. The geometrical parameters were estimated via pixel-based analysis of high-resolution images, calibrated against a known reference area.

Prototype	P1	P2.1	P2.2
Big loop area (μm^2)	39286	33221	32981
Small loop area (μm^2)	9606	8141	8141
Ratio from area measurement	4.090	4.081	4.051
Planned ratio	4.079	4.070	4.070
Ratio from fit	3.993 - 4.005	4.151(2)	4.092(2)
Period magnetic field (μT)	35.8	1.7903	2.8072
Gradient factor	> 572	26.81	43.15

First Prototype

From Table 5.4 we see that the area ratio of the first prototype turned out to be very close to an integer value. While this leads to a very large effective period, it also significantly increases the gradient sensitivity, as discussed in Sec. 3.1.2. For the first prototype, this creates a structural limitation, since the design is inherently susceptible to field gradients due to the large separation of approximately $120\text{ }\mu\text{m}$ between the centroids of the two SQUID loops. The effect is illustrated schematically in Fig. 5.12 in the form of a specific example. We assume a magnetic dipole source located 40 cm away from the sensor. The dipole moment is chosen such that the magnetic field at the sensor position is $5\text{ }\mu\text{T}$. However, the field is not spatially uniform; the effective magnetic field experienced by the two SQUID loops differs slightly due to the finite gradient across their separation. In this example, the difference is on the order of 5 pT.

For the offset-field measurement, this small gradient is amplified according to Eq. 3.16. Using the parameters of the first prototype, this results in an effective measurement deviation of

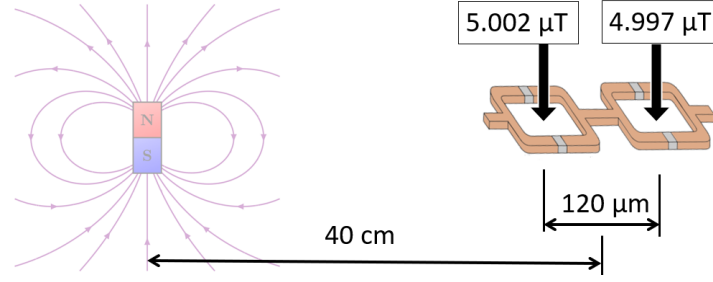


Fig. 5.12 Schematic picture of gradient sensitivity of the first prototype. A dipole source at a certain distance from the sensor is assumed. Further explanation in the main text.

approximately

$$\Delta B_{\text{meas}} \approx 2.6 \mu\text{T}.$$

Thus, when measuring a $5 \mu\text{T}$ field due to a dipole at a distance of 40 cm, the readout may yield values between roughly $2.4 \mu\text{T}$ and $7.6 \mu\text{T}$ depending on the orientation of the sensor relative to the gradient.

More homogeneous magnetic field sources would produce smaller errors, while strongly inhomogeneous sources would lead to even larger deviations. Due to this strong gradient sensitivity, it was not possible to perform reliable offset-field measurements with the first prototype when using only the double-layer mu-metal shield. An estimate of the offset field could, however, be obtained when using a triple-layer shield configuration. From this measurement we extract an offset field of

$$B_{\text{off}} = 0.4(2) \mu\text{T}.$$

The comparatively large uncertainty arises from the strong amplification of residual gradients discussed above.

Second Prototype

For the second prototype, the two slightly different versions were mounted simultaneously in the waveguide, separated by approximately 10 mm. The idea was to compare both measurements under identical conditions and thereby obtain more reliable conclusions about the magnetic field.

Before discussing the offset-field measurements, we first investigate gradient effects for this prototype. The design is intended to be first-order insensitive to magnetic field gradients. However, fabrication tolerances and spatially non-linear magnetic fields may still introduce an effective gradient.

To quantify this, we compare a fluxmap generated with the normal sweep coil to one generated with the dedicated gradient coil, shown in Fig. 5.3(c) on top of the waveguide. The extracted area ratios are $k_n = 4.0929$ for the normal coil and $k_g = 4.0969$ for the gradient coil. Assuming that the gradient of the normal coil is negligible compared to that of the gradient coil, the true geometric area ratio is given by k_n . The effective magnetic-field gradient factor R_{grad}^B (defined

in Eq. 3.8) can then be approximated by

$$R_{\text{grad}}^{\text{B}} \approx \frac{k_{\text{g}}}{k_{\text{n}}} = 1.00098. \quad (5.5)$$

In our case, $R_{\text{grad}}^{\text{B}}$ contains contributions from both non-linear spatial field variations (the design is only insensitive to spatially linear magnetic-field changes) and actual geometrical misalignment. To interpret this value, we assume that $R_{\text{grad}}^{\text{B}}$ originates purely from a spatial displacement between the effective centers of the two SQUID loops.

The idea is then to map the gradient factor onto an effective center-to-center loop distance d_{eff} under this assumption. This approach is not ideal: if the procedure were repeated for different magnetic fields, the estimated effective loop distance d_{eff} would increase with the relative magnetic-field gradient (more precisely with non-linear spatial field variations, which are correlated with the gradient). Nevertheless, the estimated d_{eff} can be used as an upper bound for fields with smaller relative gradients than the reference field. This is appropriate here, since the gradient coil produces a strong gradient compared to typical external magnetic-field sources located farther away.

Using the Biot–Savart law together with the known orientation and position of the gradient coil, we estimate an effective center-to-center offset of

$$d_{\text{eff}} \approx 4.3(1.5) \mu\text{m},$$

where the uncertainty propagates from the uncertainty of the exact position. This estimate can be regarded as an upper bound and used to assess gradient-induced measurement errors. Any realistic external magnetic-field source is expected to generate a more linear field distribution across the device area.

We now revisit the dipole example shown in Fig. 5.12, where a magnetic dipole located 40 cm away produces a field of $5 \mu\text{T}$ at the sensor. According to Eq. 3.16 and using $d_{\text{eff}} = 4.3 \mu\text{m}$, the resulting deviation in the measured field is estimated to be $\Delta B_{\text{meas}} < 7.7 \text{ pT}$. This lies in the pT range (in contrast to the μT range for the first prototype) and demonstrates the significant improvement in gradient robustness achieved by the improved design, in which the loops share a common center and the area ratio is farther from an integer value.

For this cooldown, a small single-layer mu-metal shield was used. The magnetic field inside the shield was measured independently at room temperature using a Lakeshore F71 multi-axis teslameter with a Hall sensor. Due to the mounted waveguide, the exact position of the sensor chip could not be accessed directly. In the accessible region, the measured field varied between 0.3 and $0.5 \mu\text{T}$. Higher fields (up to $25 \mu\text{T}$) were only detected very close to the SMA connector visible in Fig. 5.3(b) on top of the coupler. This field, however, decayed rapidly with distance and is not expected to generate fields above $0.5 \mu\text{T}$ at the sensor chip location. The Hall sensor is not designed for measurements at such low field levels, and the value of $0.5 \mu\text{T}$ lies close to its lower detection limit. The reliability of these measurements is therefore limited.

To extract the offset field from the fluxmaps, we identified all possible candidate points where the zero-flux condition for both loops could be fulfilled. Ideally, this corresponds to the lower mode

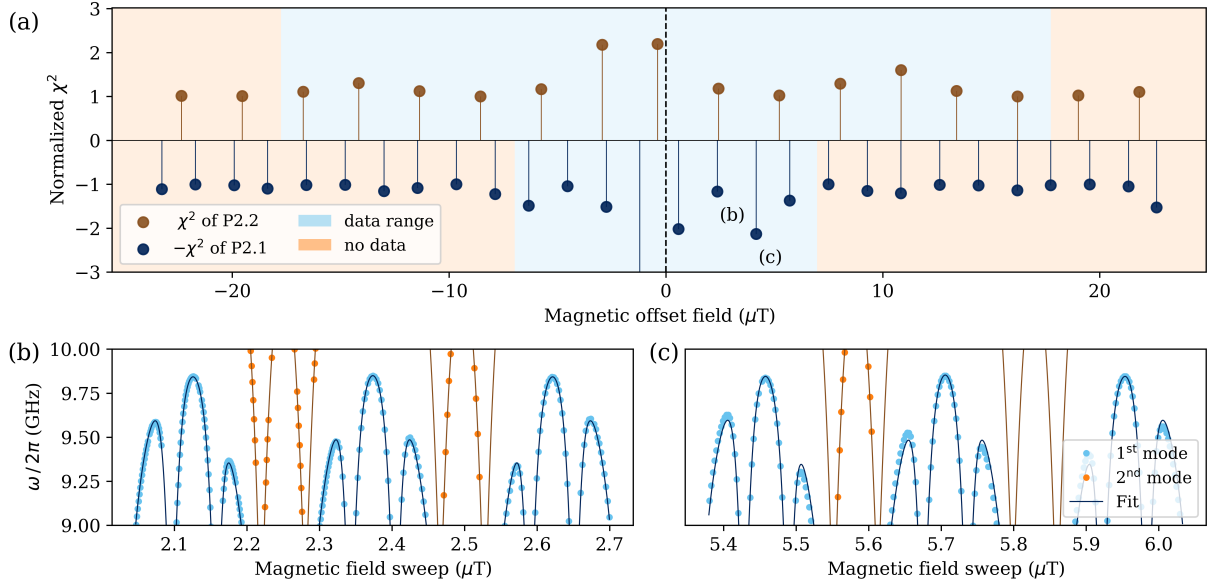


Fig. 5.13 Potential offset fields of two versions of the second prototype. (a) The offset candidates with their relative Chi-squared value, the values for the second sensor are multiplied by (-1) . The two fits indicated are shown in (b) and (c). (b) Fit with a relatively low Chi-squared value, the offset field in this fit is set to $2.37 \mu\text{T}$. (c) Fit with a relatively high Chi-squared value, the offset field of this fit function is set to $5.71 \mu\text{T}$.

exhibiting one central peak with two side peaks of equal height, as illustrated in Fig. 5.13(b). Since residual gradients can slightly break this symmetry, we also considered points where the side peaks are nearly equal.

Instead of manually comparing peak heights, we used a more systematic approach: for each potential offset candidate, the full fit function was evaluated with an adjusted offset-field parameter, and a local optimisation was performed to minimise the loss between data and model. Because the fit function is intrinsically symmetric, increasing asymmetry of the side peaks leads to a higher χ^2 value. In this way, we assess the plausibility of each candidate by comparing their relative χ^2 values (see Fig. 5.13(a)). Fits with lower χ^2 correspond to more symmetric side peaks and are therefore more likely to represent the true offset field. Two example fits are shown in Fig. 5.13(b) and (c), illustrating a low- χ^2 and a high- χ^2 case, respectively.

In the specific example shown in Fig. 5.13, the side peaks differ in height by approximately 25 MHz. From this asymmetry, we can estimate the deviation from the exact zero-flux condition at the position of the central peak, which would be fulfilled only if the side peaks were perfectly symmetric. For the prototype considered here, a frequency difference of 25 MHz corresponds to a flux imbalance of $0.014 \Phi_0$ in the larger loop or $0.0035 \Phi_0$ in the smaller loop. If the central peak is assumed to represent the nominal zero-flux condition, this value provides an estimate of the effective flux offset introduced by residual magnetic-field gradients.

The results shown in Fig. 5.13(a) are not fully conclusive with respect to the expected field magnitude. Two interpretations are possible:

1) Homogeneous-field assumption. If the field inside the shield is approximately homogeneous, both sensors (separated by only 10 mm) should measure nearly identical offset fields. This yields

$$B_{\text{off}} = 2.42(4) \mu\text{T} \quad \text{and} \quad 2.37(4) \mu\text{T}$$

for the two devices, where the error bar comes from a 1.5 % uncertainty in the area measurement. This points are shown in Fig. 5.13(a) as the dark blue point marked with (b) and the brown point opposite. The room temperature Hall-sensor measurements support the assumption of spatial homogeneity over distances larger than 10 mm. Furthermore, these candidates correspond to comparatively low χ^2 values. However, the extracted field magnitude is significantly larger than expected. If the prototype measurements are correct, this would imply that the effective permeability of the shield material decreases to roughly 20 % of its room-temperature value.

2) Small field with a large gradient. Alternatively, one could assume that the absolute field is small but contains a sizable gradient. This interpretation yields

$$B_{\text{off}} = 0.583(9) \mu\text{T} \quad \text{and} \quad -0.384(6) \mu\text{T}.$$

These are the two values closest to $B = 0$ in Fig. 5.13(a) and are closer to the expected magnitude. However, the nearly $1 \mu\text{T}$ difference between two devices separated by only 10 mm is difficult to reconcile with the Hall-sensor measurements. In addition, these candidates exhibit significantly larger χ^2 values and are therefore less likely.

In both scenarios, the room-temperature estimation of the offset field appears to be inconsistent with the results of the cryogenic measurements. It should be noted that the Hall-sensor measurement was performed one week after the experiment. Although we attempted to reproduce the experimental conditions (location, orientation, same shield, same waveguide), unavoidable differences such as changed cabling and different sample chips in the waveguide may have affected the field distribution. It should further be noted that the Hall sensor is not designed for measurements at such low field levels. Changing the orientation of the sensor at the same position, for example, resulted in reproducible variations between $0.3 \mu\text{T}$ and $0.6 \mu\text{T}$. The reliability of the Hall-sensor measurements in this regime is therefore questionable. It is possible that the room-temperature estimate of the offset field is inaccurate, rather than the cryogenic measurement.

An additional source of uncertainty is the temperature dependence of magnetic materials. It has been reported that the permeability of conventional mu-metal decreases significantly at cryogenic temperatures [49, 50]. Alloys specifically designed for cryogenic applications are typically used in superconducting experiments to avoid this issue. The shielding performance changes, but does not drop heavily at low temperatures [51]. The small single-layer shield used here is the oldest shield in the laboratory. Its exact material composition is unknown but most certainly it is made out of an alloy optimized for shielding at low temperature. We performed test measurements with the Hall sensor at liquid nitrogen temperature ($\sim 77 \text{ K}$) to assess possible changes in the shielding performance. No significant difference compared to the room-temperature measurements was observed. While further changes at lower temperatures cannot be excluded, this result makes a strong degradation of the shielding performance less likely.

5.3.2 Attenuation of the shield

To quantify the magnetic-field attenuation of the double-layer Mu-metal shield, we compared the magnetic field change measured by the sensor to an externally generated magnetic field. For this measurement, the first prototype was used together with a $5 \times 10 \times 20$ mm neodymium permanent magnet. The magnet was placed outside the shield (and cryostat) and manually moved to vary the magnetic field at the sensor position.

As a first step, the magnet was characterised using a conventional Hall sensor capable of measuring fields up to 200 mT along one spatial axis. Since the distance between the magnet and the sensor was always significantly larger than the spatial dimensions of the magnet, the magnet can be approximated as a magnetic dipole. We performed 190 independent measurements for different magnet orientations and distances. From these measurements, the dipole moment of the magnet was estimated to be $m = 1.10(4) \text{ A m}^2$. Knowing the dipole moment and the spatial position of the magnet relative to the double-SQUID device inside the cryostat allows us to estimate the magnetic field at the sensor position in the absence of shielding.

To determine the field inside the shield, we measured the resonance-frequency shift induced by the external magnet. If the magnetic-field change is sufficiently small, one could estimate the field variation from a single operating point in the fluxmap by converting frequency shifts using the local slope. However, for larger field changes this method becomes unreliable, particularly near extrema where the slope vanishes or when the resonance leaves the measurement window. Therefore, we used a more robust method: for each magnet position, we recorded 8 VNA frequency sweeps while slightly varying the coil current inside the shield. The maximum applied coil current was approximately 2 mA, corresponding to roughly one flux quantum in the SQUID loops. The fluxmap model was then fitted to these 8 data points. In this fit, all circuit parameters were fixed to values previously determined from a full fluxmap; only the offset field was treated as a free parameter. That way, there are always several points with different slopes in the measurement. This ensures that the field change can be tracked reliably. The measurement results are shown in Fig. 5.14. A clear correlation between the estimated external field and the measured internal field shift is observed. Although the overall trends agree, the detailed shapes differ slightly. This deviation is expected, since the shield geometry modifies the field-line distribution based on position. Even if the Hall sensor measures only one field component, the shielding process depends on the full three-dimensional field configuration.

A linear fit to the data yields an average attenuation factor of

$$\frac{B_{\text{outside}}}{B_{\text{inside}}} = 250(17) \hat{=} 48.0(6) \text{ dB}.$$

It should be noted that a magnetic dipole at finite distance produces a spatially inhomogeneous field at the sensor location. Depending on the relative orientation between the magnet and the device symmetry axis, the measurement may be influenced by residual gradient sensitivity of the prototype. In the optimal configuration, the magnet is oriented at 90° relative to the device's gradient-sensitive axis, minimizing gradient-induced errors. In the worst case (0° alignment), the gradient component along the sensitive axis would be maximal. During the measurement, the angle between the dominant field gradient and the sensor axis varied between 71° and 75° as the magnet was moved closer to the cryostat. Since we are primarily interested in differential

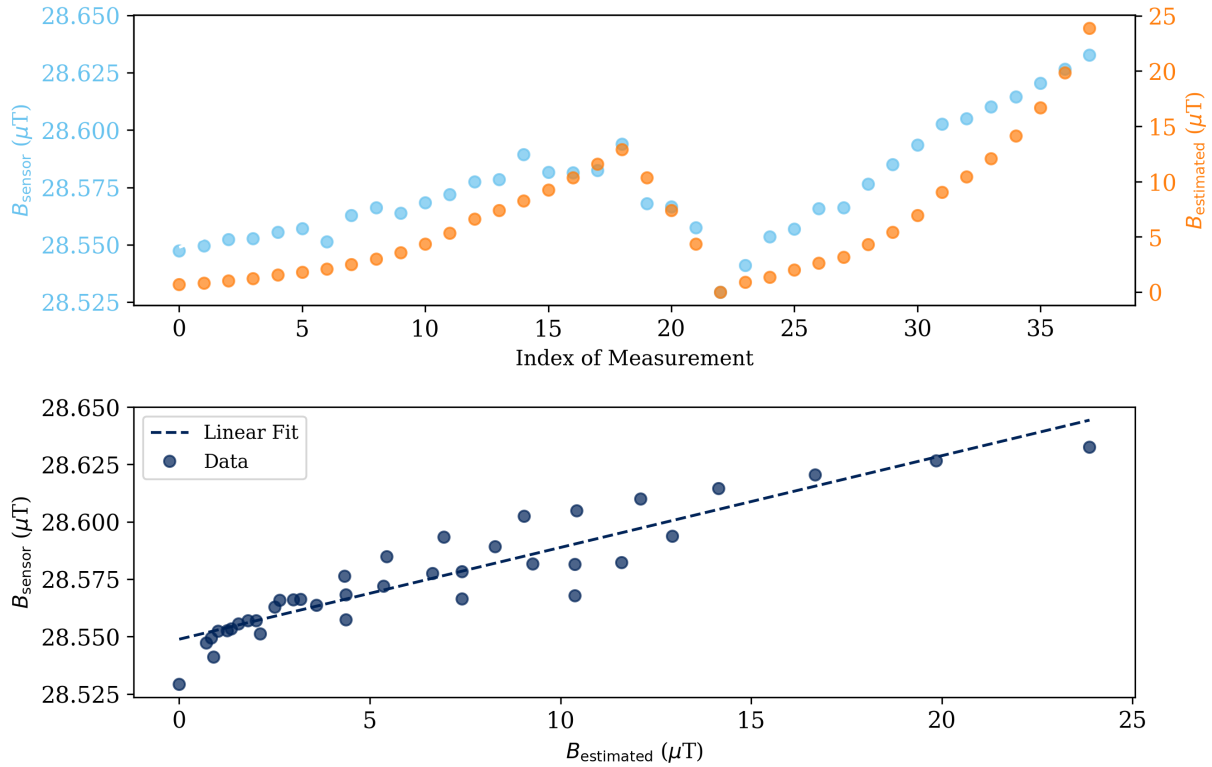


Fig. 5.14 Field measurements using a permanent magnet. Top: The magnetic field shift extracted from the sensor is plotted (left axis, blue). For each magnet position, the expected magnetic field at the sensor location is calculated using the dipole model. Bottom: Measured field shift versus expected field. A linear fit is used to estimate the attenuation factor of the shield.

field changes rather than absolute field values, the impact of this gradient is significantly smaller than in the offset-field measurements discussed earlier.

The field inhomogeneity mainly introduces small distortions in the fluxmap shape, meaning that the model used to extract the offset field does not perfectly describe the measured points. However, because the induced field changes were limited to approximately one flux quantum in the larger loop, these effects remain small. For substantially larger field excursions, gradient-induced deviations would become more significant and would need to be explicitly included in the model.

The measured attenuation is significantly smaller than the value calculated from Table 5.1. Still, the measured value is plausible, since the formula assumes infinitely long cylinders. It does not account for the critical points of the shield, which are the sharp edges at the top and bottom as well as leakage through gaps in the lid. The shielding performance also directly scales with the magnetic permeability of the material, which varies with the applied magnetic field strength. In the calculation shown in Table 5.1, a constant (standard) value for the permeability was assumed.

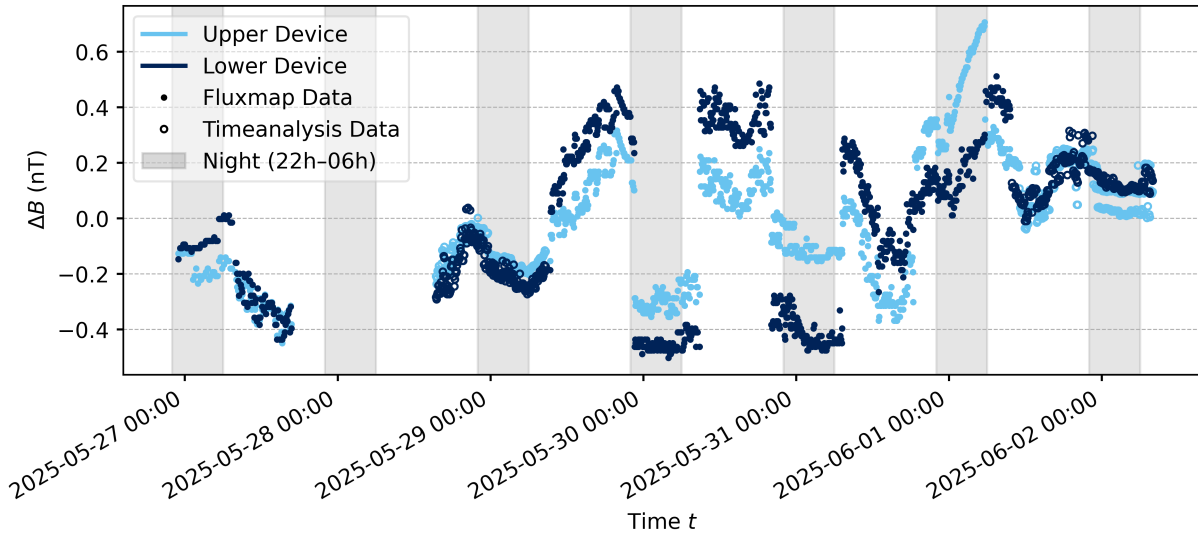


Fig. 5.15 Long-term magnetic-field measurement. The measurement was performed on two different devices placed inside the same waveguide (indicated by different colours). Two different measurement types are distinguished by markers. Three notable features are visible: (1) a clear difference between day and night behaviour, (2) four sudden jumps between 29.05 23:00 and 31.05 10:00, and (3) strong shifts on 31.05 and 01.06.

5.3.3 Magnetic field change over time

Another investigation concerned the temporal stability of the magnetic field. We analysed both long-term field drifts and the noise characteristics of the system.

Long-term changes

To investigate long-term magnetic-field variations, we repeatedly measured a frequency sweep at a fixed, flux-sensitive operating point. This slice was defined as a “reference” measurement. The chosen point lies on a steep part of the fluxmap to maximise sensitivity to small magnetic-field variations.

On the one hand, it was part of dedicated noise-characterisation measurements (discussed later in this chapter). On the other hand, it was integrated into regular fluxmap scans. During those scans, independent of the coil current used at that moment, every n^{th} sweep the coil current was tuned back to the reference point and a single VNA trace was recorded. In practice, this resulted in one reference measurement approximately every 4 minutes.

The collected data spans several days and is shown in Fig. 5.15. One clear observation is that the magnetic noise level is lower during the night. Several potential noise sources exist in the laboratory environment that are primarily active during daytime operation. By correlating the measured field fluctuations with external signals, one identifiable source was the movement of the elevator in the building.

A second conspicuous feature is the presence of discrete jumps in the measured field. The four

large jumps between 29.05 and 31.05 coincide with large changes in the coil-current operating point of the fluxmap measurement. Each fluxmap in this time window required approximately 11 hours. The first and third fluxmaps were recorded at strongly negative coil current, while the second and fourth were taken at strongly positive coil current.

The comparison between the two devices, which were separated by approximately 10 mm inside the same waveguide, shows that the magnitude of the shift differs slightly between them. This indicates a spatial dependence of the hysteresis effect, suggesting that magnetisation of nearby components contributes locally. This observation motivated the dedicated hysteresis investigation presented in Sec. 5.2.4.

The third notable feature is the strong drift visible in the final third of the dataset. This period coincides with a temperature change inside the cryostat from 30 mK to 40 mK. Such a temperature increase may influence the resonance frequency directly through changes in kinetic inductance or quasiparticle population, without necessarily reflecting a true change in magnetic field. From the available data, it is not possible to distinguish unambiguously between a genuine magnetic-field drift and a temperature-induced frequency shift. For the characterisation and offset field measurements, temperature changes have only a minor effect. Still, care was taken to eliminate any effects of temperature drifts in the offset field measurements of the second prototype by monitoring the temperature of the waveguide using a temperature sensor mounted on it.

Noise characterisation

In order to analyse the capabilities of the magnetic field sensor, we estimate the noise equivalent magnetic field $S_B^{1/2}$. This quantity is a measure of the detection sensitivity of the device and includes both the internal susceptibility and the magnetic field noise amplitude from the environment.

To investigate short-time changes, we modified the measurement method to achieve a higher sampling rate. For this purpose, we used a continuous tone measurement probing exactly at the resonance with a constant input power and averaged the signal over a short period to obtain a data point. A higher sampling rate of the scattering parameter increases the ability to resolve fast changes in the spectrum, but at the same time lead to more noise in the data, since the averaging time per point decreases. Increasing the input power to reduce the noise is not possible in our case, as this leads to transition saturation, as discussed in Sec. 5.2.3 and thus decreases the cavity response.

The sampling rate was chosen to be 200 Hz over a period of $T = 200$ s, delivering $N = 40\,000$ data points per measurement with an input power of $P_{\text{in}} = -60$ dBm, as this provided an acceptable noise level for each point. In order to map the response of the signal to a frequency change, a frequency sweep is performed before the continuous-tone measurement starts. A circle-fit is applied to this data. The fit result is a circular shape in the complex plane, where each point is parameterized by a frequency. Under the assumption that the shape of S_{21} remains constant, any measurement point in the complex plane can be mapped to a frequency by finding the geometrically closest circle-fit point. One example of this procedure is shown in Fig. 5.16. From

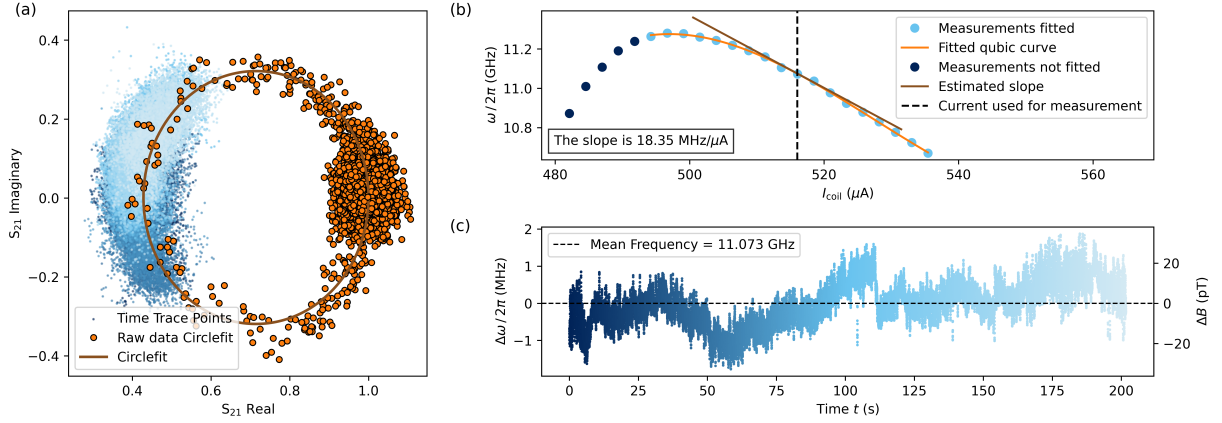


Fig. 5.16 Continuous tone measurement. (a) For every measurement, a frequency sweep is performed beforehand. The response of this sweep is fitted with a circle-fit. After the sweep, a single measurement tone is applied at constant frequency. (b) Detail of the fluxmap where the data was taken. The slope is estimated from a local cubic fit. (c) The response of the constant tone measurement can be mapped onto a frequency value using the circle-fit. The frequency shift can be converted to a magnetic field using the fluxmap slope and the current calibration.

the frequency change, we can derive the change in magnetic field using

$$B(t) = \omega(t) \cdot \frac{\partial B}{\partial \omega} = \omega(t) \cdot \frac{\partial I_{\text{Coil}}}{\partial \omega} \frac{\partial B}{\partial I_{\text{Coil}}}, \quad (5.6)$$

where the slope of the resonance curve from the flux map, $\partial\omega/\partial B$ is needed, as well as the conversion from coil current to magnetic field, $\partial B/\partial I_{\text{Coil}}$, obtained from the device calibration. The operating point together with the local slope is shown in Fig. 5.16(b). We calculate the noise equivalent magnetic field using the normalized discrete Fourier transform

$$S_B^{1/2} = \frac{\mathcal{F}\{B(t)\}}{\sqrt{BW}\sqrt{8}}, \quad (5.7)$$

with the bandwidth $BW = 1/(2T)$. The Fourier transform is normalized by dividing by the number of measurement points N . Furthermore, an additional normalization factor of $1/\sqrt{8}$ is included to account for the conversion from peak-to-peak to root-mean-square values and for the fact that only positive frequencies of the symmetric DFT spectrum are considered. With this normalization, the transformation is consistent with the values reported in [10]. Additionally, we calculate the noise equivalent magnetic flux

$$S_\Phi^{1/2} = S_B^{1/2}/A_{\text{loop}}$$

using the larger loop area $A_{\text{loop}} = 39286 \mu\text{m}^2$ of the device. The Fourier transform was performed separately for each of the 334 measurements, and the resulting spectra were averaged to obtain a reliable estimate of the noise equivalent magnetic field. The result is shown in Fig. 5.17. Several low-frequency noise peaks are visible in the spectrum. The origin of these peaks is not fully understood. One possible explanation is the pulse-tube cooler, as it operates at low frequencies. In 12 measurements, the pulse-tube was switched off; however, the spectrum did not show a

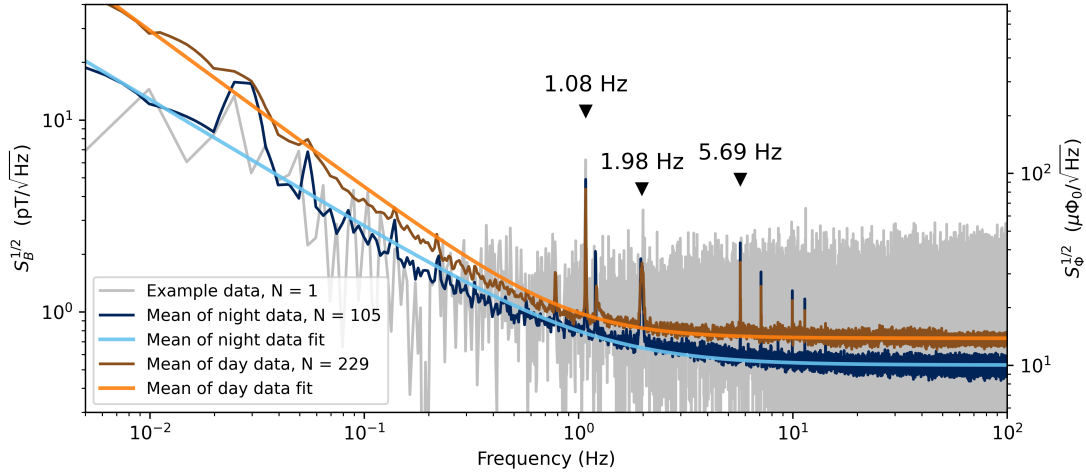


Fig. 5.17 Fourier transform of the time-domain measurement of the first design with Mu-metal shielding inside the cryostat. In gray, a single measurement is shown. Light blue represents the average over several daytime measurements, dark blue includes only nighttime measurements.

significant difference. Nevertheless, we cannot completely rule out the pulse-tube as a source, since a second cryostat in the same laboratory was running continuously.

It can also be seen that the noise floor is significantly lower during the night over the entire measured frequency range. This indicates that, at least during the day, the noise is limited by magnetic fields originating outside the can rather than by the measurement method itself.

We fitted the noise model

$$S_{\Phi}^{1/2}(\omega) = \sqrt{\frac{a}{\omega^{\alpha}} + S_0}, \quad (5.8)$$

which includes $1/f$ noise with variable exponent α as well as a white noise amplitude S_0 . From this fit, we could extract the white noise amplitude to be

$$S_{0,\text{day}}^{1/2} = 13.8 \mu\Phi_0/\sqrt{\text{Hz}} \quad \text{and} \quad S_{0,\text{night}}^{1/2} = 10.0 \mu\Phi_0/\sqrt{\text{Hz}}$$

during the day and at night.

Measurement with second prototype

With the second prototype, we performed the noise characterisation in a similar way as for the first one. As mentioned previously, in this cooldown no magnetic shield was installed inside the cryostat. Instead, the shield was placed outside the fridge in the form of a single-layer mu-metal shield. This shielding is weaker compared to the other configurations and operates close to the sensitivity limit of the sensor.

Nevertheless, continuous tone measurements were performed with this sensor-shield configuration. The measurement was carried out with exactly the same settings as for the first prototype (method, sampling frequency, duration, etc.) and only during nighttime. The result of the Fourier transformation is shown in Fig. 5.18.

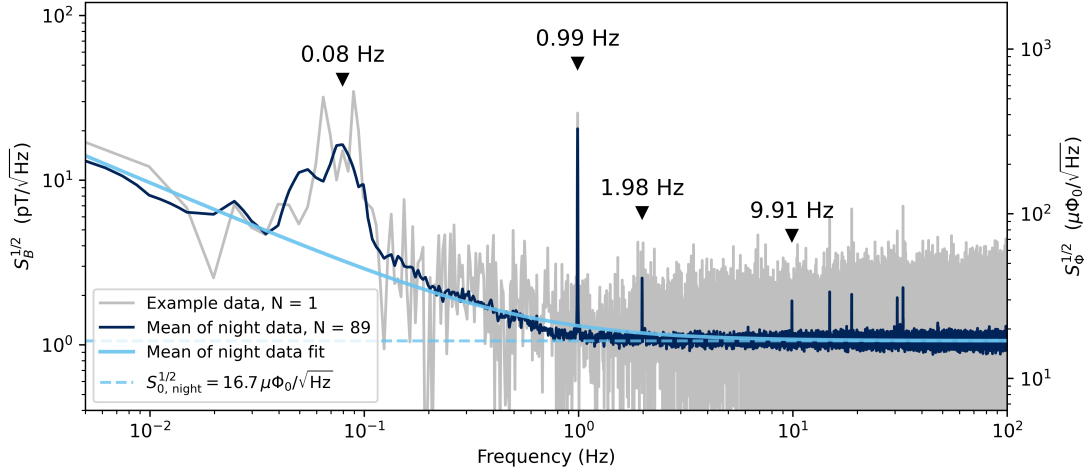


Fig. 5.18 Fourier transform of the time-domain measurement of the second design with Mu-metal shielding outside the cryostat. In gray, a single measurement is shown; in dark blue, the mean of all measurements (nighttime only).

Interestingly, a very strong and sharp peak appears at 1 Hz. In addition, a broad peak centered around 0.08 Hz is visible. This feature can also be observed directly in the time-domain data as an oscillation with a characteristic timescale of approximately 40 s. Such low-frequency noise is likely related to environmental magnetic fluctuations or laboratory infrastructure.

It should be noted that this data was taken in the Qinu SL-101-Pro cryostat in a different laboratory. Therefore, it is expected that the dominant noise sources may differ in frequency and do not necessarily coincide with those observed for the first prototype. We fitted the spectrum with the noise model from Eq. 5.8 extracted a white-noise amplitude of

$$S_{0,\text{night}}^{1/2} = 16.7 \mu\Phi_0/\sqrt{\text{Hz}}$$

which is slightly higher than the value measured for the first prototype, as expected for weaker magnetic shielding.

Conclusion and Outlook

The aim of this work was to develop a magnetic-field sensor based on a double-SQUID resonant circuit. The goal was to perform absolute magnetic-field measurements by choosing the loop areas such that the effective magnetic-field period is enlarged.

We investigated the behaviour of such a system, developed designs and fabricated two prototypes. We derived a theoretical model describing the dependence of the resonance frequency on the applied magnetic field. The model captured the main features of the experimental data well and enabled the extraction of effective device parameters, although some fitted values deviated substantially from the design estimates. By analysing the first device, we identified its shortcomings and develop an improved design for the second. The prototypes were well characterised and used to perform magnetic-field measurements.

Key characteristics of the devices are the field period and the gradient sensitivity. The first prototype had a large magnetic-field period of $T_{\text{comb}}^{\text{B}} = 35.8 \mu\text{T}$ due to fabrication uncertainty. This large period together with the spatial design made it extremely gradient-sensitive and thus effectively unusable. The second prototype had a magnetic-field period of $T_{\text{comb}}^{\text{B}} = 2.8 \mu\text{T}$. This small period together with the improved spatial design largely suppressed the gradient sensitivity.

For the offset-field measurement, we obtained values for two different shield configurations. The first one is $B_{\text{off}} = 0.4(2) \mu\text{T}$ for the triple-layer (mu-metal, niobium, mu-metal) shield measured with the first prototype. The error bar on this value is due to the large gradient sensitivity of this device. The second result is $B_{\text{off}} = 2.42(4) \mu\text{T}$ and $B_{\text{off}} = 2.37(4) \mu\text{T}$ for the single-layer mu-metal shield, measured with two versions of the second prototype placed close to each other. The measured field values are higher than expected based on the room-temperature measurements. However, it remains unclear whether this discrepancy arises from inaccuracies

in the room-temperature estimate or in the cryogenic measurement.

The offset-field measurements are not conclusive enough to confirm the working principle for absolute-field measurements due to insufficient shielding. The second prototype, with its improved design, performs better than the first, but it was tested only with weaker shielding. Thus the measurements were performed at the edge of the accessible measurement range and we could not fulfil the criterion that the prior knowledge of the magnetic field should be smaller than the magnetic-field period.

Besides the aspects directly related to absolute-field measurements, several additional characteristics of the prototypes were investigated. The measurements of the anharmonicity were not fully conclusive, but both prototypes were in a regime between a single-qubit regime and a weakly linear resonator. The second prototype showed a very broad linewidth with a quality factor of only $Q_{\text{tot}} = 230(7)$. The broad linewidth, however, turned out to be advantageous, as it has two effects: (1) it shifts the device behavior more toward that of a weakly anharmonic resonator, allowing higher readout power; (2) the broad resonance can be detected more easily, allowing lower frequency sampling rates. Both of these effects allow large fluxmaps to be measured in shorter time, which is an essential feature for speeding up the offset-field measurement.

We measured field changes with the first prototype. These measurements are not directly connected with the absolute-field capability of the devices and could in principle also be performed with a single SQUID loop. From these measurements we could determine the attenuation of the double-layer mu-metal shield to be 48.0(6) dB. Similarly, we characterised the magnetic-field noise of the same shield by fitting a noise model containing $1/f$ noise plus a white noise amplitude S_0 . We could determine the latter to be $S_{0,\text{day}}^{1/2} = 13.8 \mu\Phi_0/\sqrt{\text{Hz}}$ and $S_{0,\text{night}}^{1/2} = 10.0 \mu\Phi_0/\sqrt{\text{Hz}}$ during the day and at night, respectively.

The next step of the project should be to confirm the offset-field measurements of the second prototype in further experiments. This could be achieved by

- using a Hall sensor suitable for cryogenic temperatures in order to obtain reliable prior knowledge of the magnetic field
- using a shield with significantly higher attenuation to ensure that the offset field is well below $1 \mu\text{T}$ and thus smaller than the magnetic-field period

In both cases, two different prototypes should be used close to each other in order to allow a direct comparison of the results.

Next steps could also include improving the device design. One possibility would be to extend the magnetic-field periodicity by minor design changes that slightly modify the area ratio. The period could be increased by a factor of two to four while still maintaining good gradient insensitivity. Another optimisation step could be to investigate the mutual interaction of the loop currents in the two SQUID loops. Although the design is intended to minimise the mutual interaction, the effect could be further reduced by using, for example, round SQUID loops. Another design change could be to remove the capacitive elements. The decreased capacitance would then have to be compensated by an increased junction inductance, which would reduce β_L of both loops. In that case the shunt and coupling capacitances would play a more significant role in the total capacitance, resulting in the higher-frequency mode leaving the measurable

range. This second mode is not required for the operation of the device and mainly improves the accuracy of the fitting parameters; therefore, this trade-off could favour a simpler design.

In a broader perspective, such devices could also be adapted for other applications. A possible next step would be to measure the device in a more compact setup instead of a waveguide and perform a reflection measurement. In that case one could also consider placing three chips in a rectangular configuration to enable three-dimensional magnetic-field measurements.

Appendix

7.1 Solution of the circuit model with linear inductance

The derivation of the resonance frequencies for the circuit shown in Fig. 3.7 is omitted from the main text due to its length and is presented here instead. The problem of determining the resonance frequencies reduces to solving a cubic equation. The analytical solutions are summarised below in a form adapted to the notation of the circuit model. For a detailed derivation, the reader is referred to [52].

The input impedance of this system can be written as

$$Z_{\text{in}} = \frac{1}{iC_s\omega + \frac{1}{iL_g\omega + \frac{1}{iC_2\omega - \frac{i}{L_2\omega}} + \frac{1}{iC_1\omega - \frac{i}{L_1\omega}}}} - \frac{i}{C_c\omega}, \quad (7.1)$$

where the only difference to Eq. 3.18 is the additional term $iL_g\omega$. After factorising this term, the impedance can be written as

$$\begin{aligned}
Z_{\text{in}} &= \frac{N}{D} \\
N &= \omega^6 \left(C_1 C_2 C_c L_1 L_2 L_g + C_1 C_2 C_s L_1 L_2 L_g \right) \\
&\quad - \omega^4 \left(C_1 C_2 L_1 L_2 + C_1 C_c L_1 L_2 + C_1 C_c L_1 L_g + C_1 C_s L_1 L_2 + C_1 C_s L_1 L_g \right. \\
&\quad \left. + C_2 C_c L_1 L_2 + C_2 C_c L_2 L_g + C_2 C_s L_1 L_2 + C_2 C_s L_2 L_g \right) \\
&\quad + \omega^2 \left(C_1 L_1 + C_2 L_2 + C_c L_1 + C_c L_2 + C_c L_g + C_s L_1 + C_s L_2 + C_s L_g - 1 \right) \\
D &= C_c \omega \left[C_1 C_2 C_s L_1 L_2 L_g \omega^6 \right. \\
&\quad - \omega^4 \left(C_1 C_2 L_1 L_2 + C_1 C_s L_1 L_2 + C_1 C_s L_1 L_g + C_2 C_s L_1 L_2 + C_2 C_s L_2 L_g \right) \\
&\quad \left. + \omega^2 \left(C_1 L_1 + C_2 L_2 + C_s L_1 + C_s L_2 + C_s L_g - 1 \right) \right].
\end{aligned}$$

To determine the resonance frequencies, the roots of this equation must be found. In this case the problem reduces to finding the roots of the numerator N . Since the numerator only contains even powers of ω , the substitution

$$\omega^2 = \omega_{\text{sq}} \quad (7.2)$$

can be introduced. Furthermore, the capacitances C_s and C_c only appear as a sum in the expression. Therefore we define

$$C_E = C_s + C_c. \quad (7.3)$$

With these substitutions the equation reduces to the cubic polynomial

$$\begin{aligned}
f_{\text{cubic}}(\omega_{\text{sq}}) &= A \omega_{\text{sq}}^3 + B \omega_{\text{sq}}^2 + C \omega_{\text{sq}} + D, \\
A &= C_1 C_2 C_E L_1 L_2 L_g, \\
B &= -C_1 C_2 L_1 L_2 - C_E \left(C_1 L_1 (L_2 + L_g) + C_2 L_2 (L_1 + L_g) \right), \\
C &= (C_1 L_1 + C_2 L_2) + C_E (L_1 + L_2 + L_g), \\
D &= -1.
\end{aligned} \quad (7.4)$$

The resonance frequencies are obtained by solving $f_{\text{cubic}}(\omega_{\text{sq}}) = 0$. A general analytic solution for cubic equations exists and can be written as

$$\begin{aligned}
\Delta_0 &= B^2 - 3AC, \\
\Delta_1 &= 2B^3 - 9ABC + 27A^2D, \\
\gamma &= \sqrt[3]{\frac{1}{2}} \sqrt[3]{\Delta_1 \pm \sqrt{\Delta_1^2 - 4\Delta_0^3}}, \\
\zeta &= \frac{-1 + \sqrt{-3}}{2}, \\
\omega_{\text{sq},k} &= -\frac{1}{3A} \left(B + \zeta^k \gamma + \frac{\Delta_0}{\zeta^k \gamma} \right), \quad k \in \{0, 1, 2\}.
\end{aligned} \tag{7.5}$$

For positive capacitances and inductances ($L > 0$, $C > 0$), which is the case for the present circuit, the solutions $\omega = \sqrt{\omega_{\text{sq}}}$ are real numbers and thus also ω_{sq} . However, the intermediate quantities γ and ζ are generally complex. Substituting the explicit capacitances and inductances into the above expressions does not lead to a meaningful simplification and is therefore not shown here.

Since the physically relevant solutions are real, an alternative formulation can be used. For this purpose, the cubic equation is first transformed into its *depressed* form by the substitution

$$\omega_{\text{sq}} = t - \frac{B}{3A}. \tag{7.6}$$

This transformation removes the quadratic term and yields

$$\begin{aligned}
f_{\text{cubic}}(t) &= t^3 + pt + q, \\
t &= \omega_{\text{sq}} + \frac{B}{3A}, \\
p &= \frac{3AC - B^2}{3A^2}, \\
q &= \frac{2B^3 - 9ABC + 27A^2D}{27A^3}.
\end{aligned} \tag{7.7}$$

In this form the cubic equation admits a geometric interpretation of the roots. The three solutions can then be expressed using trigonometric functions as

$$t_k = 2\sqrt{-\frac{p}{3}} \cos\left(\frac{1}{3} \arccos\left(\frac{3q}{2p} \sqrt{\frac{-3}{p}}\right) - \frac{2\pi k}{3}\right), \quad k \in \{0, 1, 2\}. \tag{7.8}$$

The solutions t_k obtained above correspond to the roots of the transformed cubic equation. By reversing the substitution $\omega_{\text{sq}} = t - \frac{B}{3A}$, the values of $\omega_{\text{sq}} = \omega^2$ are obtained. The physical resonance frequencies of the circuit are therefore given by

$$\omega_k = \sqrt{\omega_{\text{sq},k}}, \quad k \in \{0, 1, 2\}. \tag{7.9}$$

For realistic circuit parameters, one of these solutions typically corresponds to a mode at significantly higher frequency than the other two. Since this mode usually lies outside the experimentally accessible microwave range, only the two lower-frequency solutions are relevant for the analysis of the measured spectra discussed in the main text.

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